VECTORIAL MECHANICS

SILBERSTEIN

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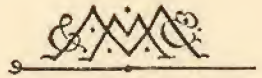

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VECTORIAL MECHANICS

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PREFACE TO THE FIRST EDITION

THE main object of this little volume is to present the chief principles and theorems of theoretical mechanics in the language of vectors, and thereby to contribute to the diffusion of the use of vectorial methods. No space has been devoted therefore to any discussion of the philosophical aspect or of the origin of such concepts as 'mass,' 'force,' 'work,' and so on, or to any description of the long and laborious route which, in the historical development of mechanics, has led to the fundamental principles of this branch of science, and especially to d'Alembert's Principle, nor of those routes which, in modern treatises, are supposed to lead to them. I emphasize d'Alembert's Principle rather than any other equivalent to it, because it is from this that a start is here made.

If our object were simply and solely a translation of mechanics into the language of vectors, we could indeed begin anywhere. But this book is not intended merely to present a loose juxtaposition of mechanical theorems and of their vectorial formulae. On the contrary, after the enunciation, in the first section of Chapter II., of the principle mentioned already we shall be able to proceed by a continuous, deductive road, so that those readers who are acquainted with little more than d'Alembert's Principle will find here an almost systematic exposition of the chief parts of mechanics. Again, other readers who know the subject thoroughly, but only in its Cartesian form, which is based on the consideration of the scalar components of displacements, velocities, forces and so on, may perhaps wish to see their knowledge translated into vectorial language, which—to say the least—is considerably shorter and more satisfactory to the imagination than the scalar language.

In this work, only such problems will be treated as are not beyond the region proper to the vector language; and consequently

everything which has no space-directional character will, with few exceptions, be omitted.

Those readers, who are already acquainted with the elementary parts of vector algebra and analysis, may proceed at once to Chapter II.; to those who are not, may be recommended 'The elements of vectorial algebra and analysis' in Chapter III., Vol. I., of Oliver Heaviside's excellent book: *Electromagnetic Theory*, London, 1893, or E. B. Wilson's *Vector Analysis*, etc., founded upon lectures of J. W. Gibbs, New York and London, 1902, or the two sections of our Chapter I., in the first of which I have endeavoured to develop the fundamental concepts of *Vector Algebra*, and in the second the most important of the *Differential and Integral Properties of Vectors*. These sections taken together will contain, I hope, all that is needed for our mechanics, and possibly also for general Mathematical Physics.

Chapters II.-VI., of which a certain part formed the subject of a series of articles by the author, published a few years ago in a Warsaw weekly (*Przegląd Techniczny*, *i.e.* 'Technical Review'), contain the *General Principles* of mechanics and their consequences, *e.g.* the three *Special Principles* and the essential part of the dynamics of *Rigid* and of *Deformable Bodies*, closing with *Hydrodynamics*.

The not very numerous collection of *Problems and Exercises* may be useful in connexion with the various chapters, and the *Appendix* containing a kind of Vectorial-Cartesian dictionary to the whole volume may be helpful to freshmen in vectorial language.

I gladly take the opportunity of expressing my sincere thanks to my friend Prof. A. M. Worthington, C.B., F.R.S., to Mr. J. F. M'Kean of the Royal Naval College, Dartmouth, to Profs. I. J. Schwatt and George H. Hallett of the University of Pennsylvania, Philadelphia, and to Profs. Alfred W. Porter, F.R.S., and R. A. Gregory, London, for their kindness in reading the MS. and the proofs, and to the Publishers for the care they have bestowed on my work.

L. S.

LONDON, *June*, 1913.

PREFACE TO THE SECOND EDITION

THIS edition differs from the first one only by the inclusion, at the end of the volume, of some Miscellaneous Notes. Two of these contain minor remarks, one gives a projective or non-metrical generalization of the concept of vector equality, another contains a concise exposition of the theory of linear vector operators, and the last gives a simplified vectorial deduction of the properties of Euler's angles determining the orientation of a rigid frame.

It is hoped that the book, apart from giving a short but, perhaps, essentially complete exposition of Mechanics, will still serve as a useful introduction to the Algebra and Analysis of Vectors and the handling of their applications.

My thanks are due to the Publishers and to the Printers, Messrs. MacLehose of Glasgow, for the care bestowed on the present edition of the book.

L. S.

ROCHESTER, N.Y., *September* 29, 1925.

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CHAPTER I.

ELEMENTS OF VECTOR ALGEBRA AND ANALYSIS.

Vector Algebra.

ANY magnitude which has *size*, in the ordinary algebraic sense of the word, as well as *direction* in space, is termed a **vector**,* whereas the common algebraic magnitudes, which have nothing to do with direction in space, which have no directional properties, but are each determined completely by a single (real) number, are called **scalars**. The typical case of a vector and, in fact, the intuitional representative of *any* vector, is a segment of a straight line of some definite length and of some definite direction† in space, the size of the vector being represented by the length, and its direction by the direction of the straight line.

Thus, the displacement of a particle from some initial to some other final position is a vector, and is represented by the segment of the straight line joining the two positions and directed from the first to the second. Other examples of vectors are the instantaneous velocity of a particle, its acceleration, its momentum, the force acting on a particle, also the instantaneous rotational velocity of, say, a rigid body round a given axis, and so on. On the other hand, mass (in classical mechanics‡), temperature, *vis viva*, energy in general; gravitational, electric or magnetic potential (of fixed

* As to the clause: 'if it obeys a certain rule of operation,' which some modern writers add to the definition of a vector, see footnote on page 4.

The term 'magnitude' is used above in the same sense as the more usual term 'quantity'; thus 'directed magnitude' stands for 'directed quantity.'

† Its *sense* being included.

‡ This reservation is necessary, as the electromagnetic mass of a moving electron, for example, has directional properties, the 'transversal mass' being generally different from the 'longitudinal mass.'

charges or magnets), mechanical or any other kind of work—are all scalars.

The size of a vector, or magnitude (absolute value) apart from direction, is called its **tensor**, or sometimes 'intensity.' Thus, the tensor of a vector is an essentially *positive scalar*.

Every vector can be determined completely by *three* scalar quantities, for instance, by its projections on any three fixed axes, orthogonal or oblique, but not coplanar,—these projections being commonly called the vector's 'components'; for example, the components of a force or the components of a velocity. We also may use polar coordinates, that is to say, we may define the tensor of a vector by the scalar r, and the direction by two other scalars, *i.e.* by two angles θ, ϵ, say the geographical latitude and longitude. In this way we get again three mutually independent scalars determining a single vector.

Obviously, such a decomposition of a vector into its three components or, more generally, into three mutually independent scalars, will in the majority of cases bring in some artificial elements, especially if the system of reference (axes, etc.) or the scaffolding constructed round the natural entities or phenomena be chosen quite at random without having anything in common with the essential characters of these entities or phenomena. Very often such a procedure gives rise to a hopeless complication of the resulting scalar formulae, a complication which does not arise from the intrinsic peculiarities of the phenomena in question, but is wholly artificial, a complication not due to Nature but to the (mathematizing) naturalist. Now, Nature is of herself wonderfully complicated; so that supplementary complication is not wanted.

This remark alone may suggest that to operate with vectors, each taken as a whole, without decomposing them into scalar components, may be more convenient and more simple, especially in those regions of research in which we are concerned *mainly with vectors* or directed magnitudes, as in Electromagnetism and in General Mechanics. But a true appreciation of the advantage of the vector method over the Cartesian (or scalar component) procedure is possible only when we see it actually at work, and the main object of each of the following chapters is to exhibit this working in Mechanics. Still more conspicuous is the service done by the vector method in Electromagnetism, especially in the hands of Oliver Heaviside, to

whom also is due that simplified form of this mathematical method, which in its main features we shall now develop.

Definition I. By saying that two vectors are *equal to one another* we mean that their *tensors* are equal and that they have the same *direction*, or, what is the same thing, that their representative straight line-segments have the same lengths and are parallel to one another and similarly (not oppositely) directed; but the equality is independent of their position in space.

According to this definition, the shifting of a given vector parallel to itself is quite immaterial, or does not change the vector.

Thus, all the vectors represented on Fig. 1 are to be considered as equal to one another. The parallel shifting of a vector, which by convention leaves it the same, is, of course, not confined to one plane.

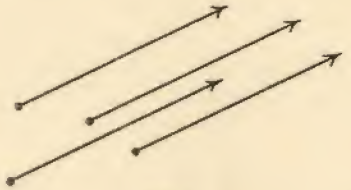

Fig. 1.

Following the example of Heaviside and Gibbs vectors will be printed in **heavy type** (**Clarendon**), and their tensors will be denoted by the same letters printed in ordinary type (or simple *italics*). Thus

$$A,\ B,\ C,\ \text{etc.},$$

will be the tensors of the vectors

$$\mathbf{A},\ \mathbf{B},\ \mathbf{C},\ \text{etc.},$$

respectively.

If the tensor of a vector, say $\mathbf{a}$, be equal to *unity* (in a given scale), *i.e.* if

$$a = 1,$$

then the vector $\mathbf{a}$ is called a **unit-vector**.

By the definition, every tensor is an absolute or *positive* number. It has, of course, the same denomination as the physical, or geometrical, quantity represented by the vector, *i.e.* if $\mathbf{A}$ be a velocity, then A signifies so many centimetres per second, and similarly in all other cases.

We pass now to the fundamental operations of vector algebra. These are: the *addition* of two vectors, and its inverse, the

subtraction of one vector from another, and two different kinds of *multiplication*, the **scalar** and the **vector multiplication** of two vectors. (The division, *i.e.* the quotient of two vectors, belongs to the Calculus of Quaternions, due to Hamilton, and has nothing to do with Heaviside's and Gibbs' vector method to be developed here, notwithstanding that the latter has grown out of the former, historically.)

Let us begin with the operation of addition and its result, the *sum* of two vectors.

Definition II. If the end of the vector **A** coincides with the beginning of another vector **B**, then we call **sum** of **A** and **B** and denote by $\mathbf{A}+\mathbf{B}$

a third vector **R** which runs from the beginning of **A** to the end of **B** (Fig. 2).*

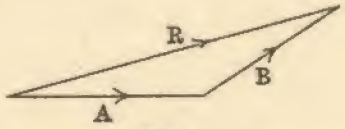

FIG. 2.

This definition of sum seems at first too narrow, as far as it appeals to the chain-arrangement of the two vectors; but in fact it embraces the concept of the sum of *any* two vectors. For, if **B** or its representative line be originally given in a quite arbitrary manner

* What we have called above a *vector*, without any further reservation, is by some modern authors called generally a *directed quantity*, whereas the term *vector* is reserved for a subclass of directed quantities. Thus, Prof. Love in his *Theoretical Mechanics* (2nd edition, Cambridge Univ. Press, 1906; pp. 8, 9) gives the following definition of a vector:

'A vector may be defined as a directed quantity which obeys a certain rule of operation. . . . This rule may be divided into two parts and stated as follows: (1) Vectors represented by equal and parallel lines drawn from different points in like senses are equivalent. (2) The vector represented by a line AC is equivalent to the vectors represented by the lines AB, BC, the points A, B, C being any points whatever.'

Thus, the 'rule of operation,' implied in, and being 'an essential part of the definition' of a vector, comprises our Definition I. (of equality of two vectors) and Definition II. (of vector-sum). The rule of operation is embodied in the quoted definition of a vector itself, obviously, for the sake of exclusion of such 'directed quantities,' as the (finite) rotation about an axis, which do not obey that rule (and which, therefore, according to that definition, are *not* vectors).

Of course, there is no serious objection to such a definition and nomenclature. But, for didactic and other reasons, it seemed to me preferable to adopt for the

relatively to **A**, we can always shift it parallel to itself (which is allowed by Definition I.) till its beginning is brought into coincidence with the end of **A**.

For the same reason we see that the sum of two vectors **A**, **B** starting from the same origin O is given by the *diagonal* OP of the *parallelogram* constructed on the addends **A**, **B** (Fig. 3). For, by Def. I., $\mathbf{B}'=\mathbf{B}$, since $B'=B$ and $\mathbf{B}'\uparrow\uparrow\mathbf{B}$ (*i.e.* $\mathbf{B}'$ parallel to and concurrent with **B**), in Euclidean space, of course.

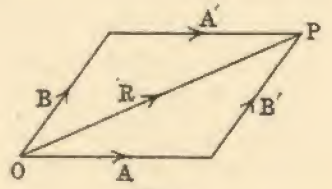

FIG. 3.

Again, in the same parallelogram, $\mathbf{A}'=\mathbf{A}$ (since $A'=A$ and $\mathbf{A}'\uparrow\uparrow\mathbf{A}$), and therefore $\mathbf{B}+\mathbf{A}=\mathbf{B}+\mathbf{A}'=\mathbf{R}=\mathbf{A}+\mathbf{B}'=\mathbf{A}+\mathbf{B}$;

hence, for any two vectors,

$$\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}.$$

present volume the usual definition (vector = directed quantity or directed magnitude), leaving it an open question, to be decided in each particular case of application, whether the parallel shifting of a vector is or is not physically indifferent and whether the sum of two vectors $\mathbf{A}+\mathbf{B}$, taken according to Def. II., does or does not represent the physical resultant or the joint effect of the two agents or 'quantities' represented separately by **A** and **B**. Thus, if **A**, **B** represent two rotations of a rigid body, namely through the angles A, B and about the axes **a**, **b**, respectively, we will still call them vectors and $\mathbf{A}+\mathbf{B}$ their vector-sum, though this last vector does *not* represent the resultant rotation of the body due to the first rotation followed by the second, nor the second followed by the first (unless the axes are coincident or the angles of rotation infinitesimal). And similarly in other cases. This can hardly give rise to misunderstanding.

Notice that if we were to introduce the more restricted definition (embodying the 'rule of operation') and if we wished to be rigorously consistent, we would find the same difficulties with regard to scalars or ordinary quantities. Thus, energy, say the electrical energy of a given field, would not deserve the name of a scalar; for, if U_1, U_2 be the energies of two electrical fields, U_1+U_2 does not represent the energy of the resultant field obtained by their superposition (unless the fields 1, 2 are mutually perpendicular). The same remark could be applied to the mass, say, of an electron at rest relatively to the observer (when it is deprived of directional properties); for the sum of the masses of two electrons is not equal to the mass of the pair of electrons (unless they are very far apart). Still, calling these entities scalars or (denominated) numbers, and operating on them as such, has never led to any trouble.

Now, the sum of two vectors being again a vector, $\mathbf{A}+\mathbf{B}=\mathbf{R}$, we can add to $\mathbf{R}$ any third vector, thus getting

$$\mathbf{R}+\mathbf{C}=(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{C}+(\mathbf{A}+\mathbf{B}).$$

Again, arranging $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ in a *chain*, *i.e.* so that the end of $\mathbf{A}$ is the beginning of $\mathbf{B}$, the end of $\mathbf{B}$ the beginning of $\mathbf{C}$, we see at once (Fig. 4) that

$$\mathbf{A}+\mathbf{B}+\mathbf{C}=(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C}),$$

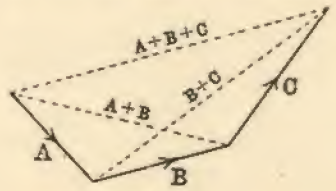

Fig. 4.

the result being always the same, namely to get from the beginning of $\mathbf{A}$ to the end of $\mathbf{C}$. The same thing is true for the sum of four, five and more vectors. Thus we get the following theorem:

Theorem I. The addition of vectors is **commutative** and **associative**, *i.e.* neither the order nor the grouping of the addends has an influence on the sum of any number of vectors.

Thus, the fundamental laws of ordinary algebraic summation of scalars hold good for vectors, without any reservation whatever.

If, in a chain-like arrangement of any number of vectors, the end of the last coincides with the beginning of the first vector, then the sum of all these vectors is *nil*. Thus, in Fig. 5,

$$\mathbf{A}+\mathbf{B}+\mathbf{C}+\mathbf{D}+\mathbf{E}=0.$$

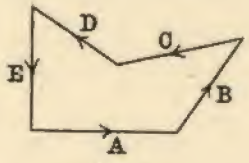

Fig. 5.

A vector is *nil* or *zero*, $\mathbf{R}=0$, if its tensor vanishes, $R=0$. In fact, this remark is scarcely necessary.

The sum of any number of vectors having the *same direction* (*i.e.* of vectors parallel and of the same sense) is a vector of the same direction. In this particular case the tensor of the sum is equal to the sum of the tensors. Thus, the common sum is a particular case of the vector-sum.

Now, let us take the case of two or more equal vectors; then we see at once that $\mathbf{A}+\mathbf{A}$ or $2\mathbf{A}$

is a vector of the same direction as $\mathbf{A}$ but of twice its tensor, *i.e.* $2A$, and that analogous properties belong to $3\mathbf{A}$, $4\mathbf{A}$, and so on. Again, understanding by $\frac{1}{2}\mathbf{A}$, $\frac{1}{3}\mathbf{A}$, etc., vectors which, repeated 2, 3, etc., times (as addends), give the vector $\mathbf{A}$, and recurring to the generally known limit-reasoning, we obtain the meaning of

$$n\mathbf{A},$$

where n is any real *positive** scalar number, whole, fractional or irrational. Thus, $n\mathbf{A}$ will be a vector which has the same direction as $\mathbf{A}$ and the tensor of which is nA. In other terms, $n\mathbf{A}$ will be the vector $\mathbf{A}$ *stretched* in the ratio $n:1$.

Thus, if $\mathbf{a}$ be a *unit-vector* having the direction of $\mathbf{A}$, remembering the definition of tensor, we may write

$$\mathbf{A}=A\mathbf{a}.$$

Any vector $\mathbf{A}$ may be represented in this way. Now A is one scalar, and $\mathbf{a}$ implies two scalars, for instance the angles θ, ϵ; thus we see again that any vector implies $1+2=3$ scalars.

The addition of two (or more) vectors may be illustrated most simply by regarding them as defining translations in space of, say, a material particle. The translation $\mathbf{A}$ carries the particle from p to p' (Fig. 6), the subsequent translation $\mathbf{B}$ carries it from p' to p''.

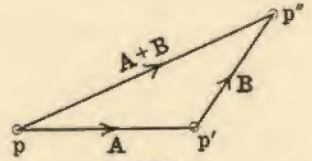

Fig. 6.

The result of $\mathbf{A}$ followed by $\mathbf{B}$, or of $\mathbf{B}$ followed by $\mathbf{A}$, *i.e.* $\mathbf{A}+\mathbf{B}$ or $\mathbf{B}+\mathbf{A}$, is to carry the particle from p to p''. Similarly, if $\mathbf{A}$, $\mathbf{B}$ be velocities of translation, $\mathbf{A}+\mathbf{B}$ will be the resultant velocity. The same applies to angular velocities, to accelerations or forces. If $\mathbf{A}$, $\mathbf{B}$ denote two forces acting simultaneously on a material

* *Negative* scalar factors n will be considered later.

particle, $\mathbf{A}+\mathbf{B}$ will be the resultant force acting on that particle. The well-known 'parallelogram law' of forces appears only as an example of the parallelogram construction (Fig. 3) and the corresponding commutative property $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$. But it must always be kept in mind that *experience only* can teach us whether any given physical quantities will or will not behave like vectors, and whether their physical 'resultant' or joined effect will be given by the sum of the corresponding vectors.

Now for the subtraction of vectors. This is the inverse of addition, and therefore will be defined precisely as in common algebra.

Definition III. We call **difference** of two vectors $\mathbf{A}$, $\mathbf{B}$ and denote by

$$\mathbf{A}-\mathbf{B}$$

such a vector $\mathbf{C}$, which added to $\mathbf{B}$ gives $\mathbf{A}$.

In other words, we say that

$$\left.\begin{aligned}&\mathbf{C}=\mathbf{A}-\mathbf{B},\\ \text{if}\quad &\mathbf{B}+\mathbf{C}=\mathbf{A}.\end{aligned}\right\}\quad\text{(Def. III.)}$$

Herefrom we see at once that if $\mathbf{A}$, $\mathbf{B}$ are arranged to be **co-initial**, *i.e.* to have the same origin O (Fig. 7), the vector $\mathbf{C}=\mathbf{A}-\mathbf{B}$ runs

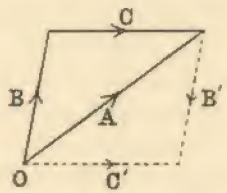

Fig. 7.

from the end of $\mathbf{B}$ *to the end of* $\mathbf{A}$. Remembering what has been said above, we see now that

$$\mathbf{A}+\mathbf{B}\quad\text{and}\quad\mathbf{A}-\mathbf{B}$$

represent the *two diagonals* of the parallelogram which is constructed upon $\mathbf{A}$ and $\mathbf{B}$ as sides, according to Fig. 8.

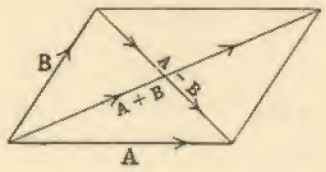

Fig. 8.

Again, from the parallelogram construction (Fig. 7), we see that

$$\mathbf{C}=\mathbf{C}'=\mathbf{A}+\mathbf{B}',$$

where $\mathbf{B}'$ has the same tensor as $\mathbf{B}$ but the *opposite* sense. This gives us another simple rule for constructing the difference of two vectors.

Comparing the last equation with the above definition, we have

$$\mathbf{A}+\mathbf{B}'=\mathbf{A}-\mathbf{B},$$

so that $-\mathbf{B}$ means the same thing as $+\mathbf{B}'$, *i.e.* the *opposite** of $\mathbf{B}$.

The same thing follows also immediately from the above definition (Def. III.) written out for the particular case $\mathbf{A}=0$; for then we have

$$\mathbf{C}=0-\mathbf{B}=-\mathbf{B}\quad\text{and}\quad\mathbf{B}+\mathbf{C}=0;$$

hence

$$\mathbf{B}+(-\mathbf{B})=0,$$

that is to say, $-\mathbf{B}$ is the vector which runs *from the end of* $\mathbf{B}$ *to its origin*, and this is precisely what has been called the "opposite" of $\mathbf{B}$.

Thus, for any two vectors $\mathbf{A}$, $\mathbf{B}$,

$$\mathbf{A}-\mathbf{B}$$

means the same thing as $\mathbf{A}+(-\mathbf{B})$,

and the subtraction is therefore reduced to the addition of vectors. Hence, if the scalar n be a *negative* number, then

$$n\mathbf{A}$$

means the vector $\mathbf{A}$ turned through 180°, and then stretched in the ratio $|n|:1$, or, what is the same thing, first stretched in this ratio and then turned.†

Thus the meaning of the product of a vector *by any scalar* n is fixed. The concept of such a product does not imply, in fact, any new concepts besides the vector sum or vector difference.

But now we will pass to the conception of other products, namely, of products of a vector by a vector, which are *not* reducible to the conception of the vector sum. These products are, as already remarked, of two different kinds, the *scalar-* and the *vector-product* of two vectors. Both of them, susceptible of far-reaching applications, belong, together with the vector sum (and difference), to the most fundamental concepts of Vector Method. Let us begin with the former of these two products of a pair of vectors.

*That is, $\mathbf{B}$ *turned by* 180°, in any plane passing through $\mathbf{B}$.

†$|n|$ means, generally, the *absolute value* of n.

Definition IV. The **scalar product** of a pair of vectors **A** and **B**, whose included angle is θ, is defined to be the *scalar* $AB\cos\theta$, and is denoted by **AB**.* Thus (see Fig. 9)

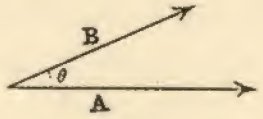

Fig. 9.

$$\mathbf{AB} = AB\cos\theta,$$

A, B being the tensors of **A**, **B**, as explained above.

We may say, equivalently, that **AB** is the component of **A** taken along **B**, multiplied by the tensor of **B**, or, what is the same thing, the component of **B** taken along **A**, multiplied by the tensor of **A**.

First of all, we have, according to the definition itself,

$$\mathbf{BA} = \mathbf{AB},$$

that is, the order of the factors in the scalar product is quite immaterial, or more concisely, the scalar product is *commutative*, precisely as an ordinary algebraic product.

If $\theta < \frac{\pi}{2}$, then **AB** is positive, and if $\frac{3\pi}{2} > \theta > \frac{\pi}{2}$, then **AB** is negative.

If **A**, **B** are *normal* or *perpendicular to one another*, *i.e.* if $\theta = \frac{\pi}{2}$, then

$$\mathbf{AB} = 0,$$

independently of the tensors of the two vectors. Conversely, if $\mathbf{AB} = 0$, then we can infer only that

$$\mathbf{A} \perp \mathbf{B},$$

and not that one of the factors vanishes. This is an important difference between the scalar product of a pair of vectors and the ordinary algebraic product.

If **A**, **B** have *opposite directions*, then $\cos\theta = -1$, and

$$\mathbf{AB} = -AB.$$

If **A**, **B** have *the same directions*, then $\cos\theta = 1$, and

$$\mathbf{AB} = AB.$$

In this particular case the scalar product of a pair of vectors degenerates into the algebraic product of their tensors.

*This is Heaviside's notation; Gibbs writes for the scalar product **A**.**B**.

If $\mathbf{B} = \mathbf{A}$, then **AB** becomes what may be called the **scalar square** of the vector **A**, and may be written

$$\mathbf{AA} = \mathbf{A}^2 = A^2.$$

Hence, if **a** be any *unit-vector*, then

$$\mathbf{a}^2 = a^2 = 1.$$

Conversely, if $\mathbf{a}^2 = 1$, we can infer *only* that **a** is a unit-vector without ascertaining anything about its direction in space.

If **a**, **b** be a pair of unit-vectors, then

$$\mathbf{ab} = \cos\theta = \cos(\mathbf{a}, \mathbf{b}).$$

Since **AB** is a scalar, its multiplication by another scalar n or by a vector **C** does not present any difficulty. Thus, $n\mathbf{AB}$ or $\mathbf{AB}n$ means the same thing as $n(\mathbf{AB})$ or $(\mathbf{AB})n$, that is n times **AB** or **AB** times n; again

$$(\mathbf{AB})\mathbf{C}$$

means **AB** times **C**, *i.e.* $AB\cos\theta\,.\,\mathbf{C}$. (The parentheses above are to be considered as separators.) But, of course, we must not confound

$$(\mathbf{AB})\mathbf{C} \text{ with } \mathbf{A}(\mathbf{BC}) \text{ or } \mathbf{B}(\mathbf{CA}),$$

and therefore the brackets (or some other separators, as dots used by Heaviside) are quite indispensable.

Observe that, **A**, **B**, **C**, **D**, etc., being any series of vectors, we have

$$\mathbf{AB} = \text{scalar},$$

$$(\mathbf{AB})\mathbf{C} = \text{vector},$$

$$(\mathbf{AB})\mathbf{CD} = \text{scalar, etc.},$$

that is to say, by introducing fresh vector factors we get alternately scalars and vectors. (This will be contrasted afterwards with the other kind of multiplication, the vector multiplication of vectors by vectors.)

In various branches of physics scalar products frequently have reference to energy, or work, or activity, *i.e.* work per unit time. Thus, if **F** be a mechanical force and **s** the displacement (say, infinitesimal) of its point of application, then $F.\cos(\mathbf{F}, \mathbf{s}).s$, *i.e.* the scalar product

$$\mathbf{Fs}$$

is the work done by the force **F**. If, **F** having the same meaning as above, **v** is the velocity of its point of application, then **Fv** is the activity of the force. Again, if **E** be the electric force and **D**

the dielectric displacement (which in anæolotropic medium is not generally identical in direction with **E**), then

$$\tfrac{1}{2}\mathbf{ED}$$

is the electric energy per unit volume. Similarly, half the scalar product of magnetic force and induction gives the magnetic energy per unit volume. In the special case of isotropy **D** is concurrent with **E**, namely

$$\mathbf{D} = K\mathbf{E},$$

the scalar K being the dielectric constant or 'permittivity' of the medium; in this case the density of electrical energy becomes

$$\tfrac{1}{2}K\mathbf{E}^2 = \tfrac{1}{2}KE^2;$$

and similarly for the magnetic energy.

Let us now pass to the scalar product of a vector **A**, and the sum of two other vectors **B** + **C**, which is fundamental in vector algebra.

By Definition IV., we have

$$\mathbf{A}(\mathbf{B}+\mathbf{C}) = A \times \text{projection of } (\mathbf{B}+\mathbf{C}) \text{ on } \mathbf{A};$$

but the projection of a sum of vectors on any direction or axis is seen immediately to be the same as the sum of the projections of the single vectors on that axis (see Fig. 10). Hence

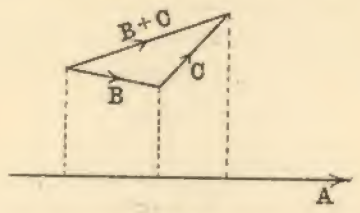

Fig. 10.

$$\mathbf{A}(\mathbf{B}+\mathbf{C}) = AB\cos(\mathbf{A},\mathbf{B}) + AC\cos(\mathbf{A},\mathbf{C}),$$

that is to say

$$\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}.$$

Similarly,

$$\mathbf{A}(\mathbf{B}+\mathbf{C}+\mathbf{D}+\ldots) = \mathbf{AB}+\mathbf{AC}+\mathbf{AD}+\ldots.$$

Thus we get the following theorem:

Theorem II. The *scalar multiplication* of pairs of vectors and of vector-sums is **commutative** *and* **distributive.**

Thus, the scalar product of a pair of vector-sums is developed precisely as in ordinary algebra; for example,

$$(\mathbf{A}+\mathbf{B})(\mathbf{C}+\mathbf{D}) = \mathbf{AC}+\mathbf{AD}+\mathbf{BC}+\mathbf{BD}.$$

Similarly, we have

$$\mathbf{A}(\mathbf{B}-\mathbf{C}) = \mathbf{AB}-\mathbf{AC},$$

since

$$\mathbf{B}-\mathbf{C} = \mathbf{B}+(-\mathbf{C}).$$

Also

$$(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B}) = \mathbf{A}^2-\mathbf{B}^2 = A^2-B^2,$$

which the reader may put easily into the form of a geometrical theorem, remembering that **A** + **B** and **A** − **B** are the diagonals of the parallelogram constructed on **A**, **B**.

Another example is

$$(\mathbf{A}+\mathbf{B})^2 = A^2+2\mathbf{AB}+B^2,$$

and, particularly, if $\mathbf{A} \perp \mathbf{B}$,

$$(\mathbf{A}+\mathbf{B})^2 = A^2+B^2,$$

which is simply the theorem of Pythagoras.

But as this book will be full of illustrations of the use of the scalar product and the remaining fundamental concepts of Vector Calculus, we have no need to look about for illustrations in this preparatory chapter.

We come now to the second kind of product of a pair of vectors, namely the *vector product.*

Definition V. The **vector product** of two vectors **A**, **B** is a third *vector* **C**, whose tensor is equal to the *area of the parallelogram* **A**, **B**, and whose direction is *perpendicular to the plane of* **A**, **B**,* the positive direction of **C** being such that a right-handed rotation about **C** through an angle less than 180° carries the vector **A** into **B** (Fig. 11). The vector product thus defined is denoted by

$$\mathbf{C} = \mathrm{V}\mathbf{AB}.$$

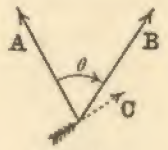

Fig. 11.

Thus, if θ be the angle included by **A**, **B**, we have

$$C = AB\sin\theta,$$

and as, by the definition, $\mathbf{C} \perp \mathbf{A}, \mathbf{B}$, we have

$$\mathbf{AC} = 0, \quad \mathbf{BC} = 0,$$

or

$$\mathbf{A}\mathrm{V}\mathbf{AB} = 0, \quad \mathbf{B}\mathrm{V}\mathbf{AB} = 0.$$

* Any two vectors can always be brought into one plane, by shifting one or the other of them parallel to itself, according to Def. I.

Also, we see by Def. V. that on interchanging the factors, the direction of **C** is reversed, its tensor remaining the same, that is

$$\mathrm{V}\mathbf{BA} = -\mathrm{V}\mathbf{AB}.$$

Thus, the vector product is *not commutative*, and the order of its factors has always to be treated with care.

Again, if **A**, **B** are *parallel* to one another, *i.e.* if $\theta = 0$ or $\theta = \pi$, then $\sin\theta = 0$, and consequently

$$\mathrm{V}\mathbf{AB} = 0.$$

Conversely, from $\mathrm{V}\mathbf{AB} = 0$ it follows only that **A**, **B**, otherwise unknown, are parallel to one another.

In particular, we have for any vector **A**,

$$\mathrm{V}\mathbf{AA} = 0,$$

which may be put in words by saying that the vectorial *auto*-product of a vector vanishes.

Again, if $\mathbf{A} \perp \mathbf{B}$, or $\theta = \frac{\pi}{2}$, $\sin\theta = 1$, we get, as the tensor of $\mathrm{V}\mathbf{AB}$,

$$C = AB;$$

thus, *caeteris paribus*, the maximum tensor is reached for $\theta = 90^\circ$.

If m, n be any pair of scalars, then, by Definition V.,

$$\mathrm{V}m\mathbf{A}n\mathbf{B} = mn\mathrm{V}\mathbf{AB},$$

just as for the scalar product

$$m\mathbf{A}n\mathbf{B} = mn\mathbf{AB}.$$

If, in particular, $\mathbf{A} = A\mathbf{a}$, $\mathbf{B} = B\mathbf{b}$,

where **a**, **b** are the corresponding unit-vectors, then

$$\mathrm{V}\mathbf{AB} = AB\mathrm{V}\mathbf{ab}.$$

If **i** **j**, **k** be a **right-handed** system of mutually *perpendicular unit-vectors*, then $\mathrm{V}\mathbf{ij} = \mathbf{k}$,

since $\sin(\mathbf{i}, \mathbf{j}) = \sin 90^\circ = 1$, and similarly

$$\mathrm{V}\mathbf{jk} = \mathbf{i}, \quad \mathrm{V}\mathbf{ki} = \mathbf{j},$$

so that the three equations follow from one another by cyclic permutation, the order

being always kept in mind as the proper order. Inverting it, we have to change the sign, thus

$$\mathrm{V}\mathbf{ji} = -\mathbf{k}, \text{ etc.}$$

Remembering the definition of scalar product, we have for the same system of unit vectors, which throughout the whole of this book will be denoted by **i**, **j**, **k**,

$$\mathbf{ij} = 0, \quad \mathbf{jk} = 0, \quad \mathbf{ki} = 0,$$

and

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 1,$$

while, of course,

$$\mathrm{V}\mathbf{ii} = \mathrm{V}\mathbf{jj} = \mathrm{V}\mathbf{kk} = 0.$$

The *components* of any vector **A** taken along **i**, **j**, **k**, *i.e.* the scalar products

$$\mathbf{Ai}, \quad \mathbf{Aj}, \quad \mathbf{Ak},$$

will be denoted by

$$A_1, \quad A_2, \quad A_3,$$

respectively; thus

$$\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}.$$

Such a decomposition will be made use of incidentally, but as a rule we shall avoid any artificial decomposition of vectors, except in transitional work or in order to show the reader the equivalent Cartesian form of some fundamental vector formula. (See also the 'Appendix' containing a great number of such equivalents, composed especially for those readers who are not used to the Vector Method.)

But, meanwhile, let us return to the vector product of any pair of vectors, say **B**, **C**. Since $\mathrm{V}\mathbf{BC}$ is a vector, it may in its turn be multiplied by a third vector, say **A**, either scalarly or vectorially, giving rise respectively to

1) $\mathbf{A}(\mathrm{V}\mathbf{BC})$ or simply $\mathbf{A}\mathrm{V}\mathbf{BC}$,

and

2) $\mathrm{V}\mathbf{A}(\mathrm{V}\mathbf{BC})$ or simply $\mathrm{V}\mathbf{A}\mathrm{V}\mathbf{BC}$.

The second of these products will be treated later.

For the present let us consider the first of these triple products, *i.e.*

$$\mathbf{A}\mathrm{V}\mathbf{BC},$$

as it will enable us to prove the most important property of vectorial multiplication, namely its *distributivity* (Theorem IV. below), without any artificial decomposition of the vectors involved.

Now, putting for the moment

$$\mathrm{V}\mathbf{BC} = \mathbf{R} = R\mathbf{r},$$

we have

$$\mathbf{A}\mathrm{V}\mathbf{BC} = \mathbf{AR}.$$

But the tensor of **R**, $R = BC\sin(\mathbf{B}, \mathbf{C})$,

is the *area* of the parallelogram **B**, **C**, which may be considered as

the basis of the parallelopiped **A**, **B**, **C** (Fig. 12), and the direction of **R** is perpendicular to this basis; hence, if the arrangement of **A**, **B**, **C** be *right-handed* (as in Fig. 12), then

$$\mathbf{AVBC} = \mathbf{A}R\mathbf{r} = R\mathbf{Ar} = \text{base} \times \text{height}$$
$$= \text{volume of the parallelopiped } \mathbf{A}, \mathbf{B}, \mathbf{C}.$$

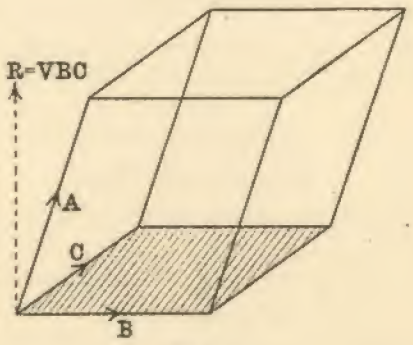

Fig. 12.

Thus the triple (scalarly-vectorial) product

$$\mathbf{AVBC}$$

has a very simple geometrical, or say stereometrical, meaning; it represents the *volume of the parallelopiped* constructed upon **A**, **B**, **C** as edges, if the order **A**, **B**, **C** be a right-handed one. But, if it be left-handed, **AVCB** represents the same thing.

Again, considering **C**, **A** as the basis, we see that **BVCA** represents the same volume, and similarly **CVAB** the same volume of the parallelopiped **A**, **B**, **C**.

Hence the following theorem:

Theorem III. *The triple, scalarly-vectorial product of any triad of vectors retains its value when its factors are cyclically permuted,*

$$\mathbf{AVBC} = \mathbf{BVCA} = \mathbf{CVAB}.$$

This theorem is of fundamental importance, in mechanical, electromagnetic, and other applications.

As the reader knows already that $\mathbf{VCB} = -\mathbf{VBC}$, it is unnecessary to emphasize that $\mathbf{AVCB} = -\mathbf{AVBC}$,

i.e. that any one of the above three products changes its sign if the cyclic order **A**, **B**, **C** be reversed.

Coming now to the capital point, let us consider the vectorial product of a vector by a vector sum or, more generally, the vector product of a pair of such sums. We then have the following fundamental theorem:

Theorem IV. *Vector multiplication is* **distributive**, *i.e.*

$$\mathbf{VA}(\mathbf{B}+\mathbf{C}) = \mathbf{VAB} + \mathbf{VAC},$$

and also $\mathbf{V}(\mathbf{A}+\mathbf{B})(\mathbf{C}+\mathbf{D}) = \mathbf{VAC} + \mathbf{VAD} + \mathbf{VBC} + \mathbf{VBD}$,

as for scalar multiplication, with the only difference that vector products are *not* commutative.

To prove this important property, let us put

$$\mathbf{VA}(\mathbf{B}+\mathbf{C}) - \mathbf{VAB} - \mathbf{VAC} = \mathbf{Q}.$$

Then we have only to show that the vector **Q** vanishes. Multiply both sides of the last equation scalarly by **A**, or **B**, or **C**; then

1) $\mathbf{AQ} = \mathbf{AVA}(\mathbf{B}+\mathbf{C}) - \mathbf{AVAB} - \mathbf{AVAC} = 0$,

since $\mathbf{VA}(\mathbf{B}+\mathbf{C})$, **VAB**, **VAC** are perpendicular to **A**; again

2) $\mathbf{BQ} = \mathbf{BVA}(\mathbf{B}+\mathbf{C}) - \mathbf{BVAC}$
$= -\mathbf{BVAC} + (\mathbf{B}+\mathbf{C})\mathbf{VBA}$, by Theor. III.,
$= -\mathbf{BVAC} + \mathbf{BVBA} + \mathbf{CVBA}$
$= -\mathbf{BVAC} + \mathbf{BVAC} = 0$,

and similarly

3) $\mathbf{CQ} = \ldots = \mathbf{BVCA} - \mathbf{CVAB} = 0$.

Thus, by 1), 2), 3),

$$\mathbf{AQ} = 0, \quad \mathbf{BQ} = 0, \quad \mathbf{CQ} = 0.$$

Hence **Q** either vanishes or is normal simultaneously to all the three vectors **A**, **B**, **C**; but if these are *not coplanar*, the last alternative is obviously impossible, so that **Q** must vanish, *i.e.*

$$\mathbf{VA}(\mathbf{B}+\mathbf{C}) = \mathbf{VAB} + \mathbf{VAC}.$$

Thus, for **A**, **B**, **C** not coplanar the theorem is proved. Again, if **A**, **B**, **C** *are* coplanar, we can always add, say to **C**, a fourth vector **D** including any angle with the plane of **A**, **B**, **C**; then **A**, **B**, **C**+**D** will not be coplanar, and therefore

$$\mathbf{VA}[\mathbf{B}+(\mathbf{C}+\mathbf{D})] = \mathbf{VAB} + \mathbf{VA}(\mathbf{C}+\mathbf{D});$$

but making **D** approach zero, we get again

$$\mathbf{VA}(\mathbf{B}+\mathbf{C}) = \mathbf{VAB} + \mathbf{VAC},$$

so that the first part of the Theorem IV. is proved for *any* triad of vectors.

To prove the second part of this theorem, *i.e.* that

$$\mathrm{V}(\mathbf{A}+\mathbf{B})(\mathbf{C}+\mathbf{D}) = \mathrm{VAC}+\mathrm{VAD}+\mathrm{VBC}+\mathrm{VBD},$$

put $\mathbf{C}+\mathbf{D}=\mathbf{R}$, and observe that

$$\begin{aligned}\mathrm{V}(\mathbf{A}+\mathbf{B})\mathbf{R} &= -\mathrm{VR}(\mathbf{A}+\mathbf{B}) = -\mathrm{VRA}-\mathrm{VRB}\\ &= \mathrm{VAR}+\mathrm{VBR} = \mathrm{VA}(\mathbf{C}+\mathbf{D})+\mathrm{VB}(\mathbf{C}+\mathbf{D}):\end{aligned}$$

then the second part of Theor. IV. is reduced to its first part; thus the whole theorem is proved.

The importance and practical applicability of vector as well as scalar products of vectors are based mainly on their *distributivity*, in which property they both resemble the common algebraic product. The scalar product is, besides, commutative (*i.e.* $\mathbf{BA}=\mathbf{AB}$), the vector product has *not* the benefit of this property in as far as

$$\mathrm{VBA} = -\mathrm{VAB};$$

but the trouble is, in fact, not very considerable; when inverting the order of the factors we have to do but a very little, namely *to change the sign* of the product, and this is very easily remembered.

In Physics, and especially in Mechanics and Electromagnetism, including Optics, the vector products are as useful as, and perhaps more than, the scalar products of vectors. The following mechanical chapters will be full of their applications. Here therefore a pair of electromagnetic examples will be sufficient.

Thus, if $\mathbf{E}$, $\mathbf{M}$ be the electric and magnetic force in any point of a field, the product

$$c\mathrm{VEM}$$

(c = velocity of light in vacuo, an ordinary scalar) gives the flux of electromagnetic energy, per unit time and per unit area, at that point. This product is generally known as the *Poynting vector*. Thus, the electromagnetic energy flows normally to the plane containing both the forces $\mathbf{E}$, $\mathbf{M}$.

Again, 'the electromagnetic force,' according to Maxwell's terminology, *i.e.* the pondero-motive force, per unit volume, acting upon a conductor supporting an electric current, placed in a magnetic field, equals the vector product of the current $\mathbf{C}$ (per unit area) and the magnetic induction $\mathbf{B}$:

$$\mathbf{F} = \mathrm{VCB}.$$

In fact, the force $\mathbf{F}$ is normal to both $\mathbf{C}$ and $\mathbf{B}$, its intensity F is given by the area of the parallelogram $\mathbf{C}$, $\mathbf{B}$, and its direction is determined by saying that $\mathbf{F}$, $\mathbf{C}$, $\mathbf{B}$ is a right-handed system.

Now, all these rules are certainly remembered more easily if condensed into the short formula $\mathbf{F}=\mathrm{VCB}$.

But let us go on with our vector algebra.

Once in possession of Theorem IV., we can immediately develop the product VAB into its Cartesian form, *i.e.* represent it by the components of its factors. In fact, if

$$\mathbf{A} = A_1\mathbf{i}+A_2\mathbf{j}+A_3\mathbf{k}$$

and

$$\mathbf{B} = B_1\mathbf{i}+B_2\mathbf{j}+B_3\mathbf{k},$$

then

$$\mathrm{VAB} = A_1B_1\mathrm{Vii}+\ldots+A_1B_2\mathrm{Vij}+\ldots;$$

but, as has been shown above,

$$\mathrm{Vii}=\mathrm{Vjj}=\mathrm{Vkk}=0,$$

and

$$\mathrm{Vjk}=\mathbf{i},\quad \mathrm{Vki}=\mathbf{j},\quad \mathrm{Vij}=\mathbf{k};$$

thus we get

$$\mathrm{VAB} = \mathbf{i}(A_2B_3-A_3B_2)+\mathbf{j}(A_3B_1-A_1B_3)+\mathbf{k}(A_1B_2-A_2B_1), \qquad (1)$$

or, in determinant form,

$$\mathrm{VAB} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\end{vmatrix}. \qquad (1a)$$

Hence, by taking a pair of unit vectors $\mathbf{a}$, $\mathbf{b}$, whose components or *direction-cosines* are a_1, etc., b_1, etc., respectively, and whose included angle is θ, we have the known trigonometrical formula,

$$(\mathrm{Vab})^2 = \sin^2\theta = (a_2b_3-a_3b_2)^2+(a_3b_1-a_1b_3)^2+(a_1b_2-a_2b_1)^2.$$

Similarly, using the distributivity of the scalar product, we have

$$\mathbf{AB} = \mathbf{i}^2A_1B_1+\ldots+\mathbf{ij}(A_1B_2+A_2B_1)+\ldots;$$

but

$$\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=1 \quad\text{and}\quad \mathbf{ij}=\mathbf{jk}=\mathbf{ki}=0;$$

hence

$$\mathbf{AB} = A_1B_1+A_2B_2+A_3B_3, \qquad (2)$$

and in particular, for a pair of unit-vectors $\mathbf{a}$, $\mathbf{b}$,

$$\mathbf{ab} = \cos\theta = a_1b_1+a_2b_2+a_3b_3,$$

which is another well-known trigonometrical formula.

Again, developing the triple product, AVBC, we get at once

$$\mathrm{AVBC} = A_1(B_2C_3-B_3C_2)+A_2(B_3C_1-B_1C_3)+A_3(B_1C_2-B_2C_1),$$

or, in determinant form,

$$\mathrm{AVBC} = \begin{vmatrix}A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\\ C_1 & C_2 & C_3\end{vmatrix}, \qquad (3)$$

which is a known expression for the volume of a parallelopiped; using this, we may verify again the Theorem III.

To complete our preparatory instruction in vector algebra one thing more is needed, namely the triple *vectorially-vectorial* product,

$$\mathbf{V}\mathbf{A}(\mathbf{V}\mathbf{B}\mathbf{C}) \text{ or, shortly, } \mathbf{V}\mathbf{A}\mathbf{V}\mathbf{B}\mathbf{C},$$

which has already been alluded to.

This product is again a *vector*, namely normal to **A** and to V**BC**; but the vector V**BC** is, by its definition, perpendicular to the plane **B**, **C**. Hence V**AVBC** *lies in the plane* **B**, **C**, and it must, therefore, be possible to represent it in the form

$$\mathbf{V}\mathbf{A}\mathbf{V}\mathbf{B}\mathbf{C} = y\mathbf{B} + z\mathbf{C}, \qquad (4')$$

where y, z are *scalars*.

The process of vector multiplication may be repeated any number of times, giving always a *vector*. In this regard vectorial multiplication differs characteristically from scalar multiplication (see the corresponding remark above). Thus V**D**(V**AVBC**) will be a vector, namely normal to **D** and to V**AVBC**, and so on.

But, in practice, we shall have to do with hardly more than triple vectors; thus, it will be sufficient here to consider only V**AVBC**.

To develop this triple product, we have only, in the determinant (1*a*), to replace B_1, B_2, B_3 by

$$B_2C_3 - B_3C_2, \quad B_3C_1 - B_1C_3, \quad B_1C_2 - B_2C_1;$$

hence the first component (*i.e.* the i-component) of V**AVBC** will be

$$\begin{aligned} &A_2(B_1C_2 - B_2C_1) - A_3(B_3C_1 - B_1C_3) \\ &\quad = (A_1C_1 + A_2C_2 + A_3C_3)B_1 - (A_1B_1 + A_2B_2 + A_3B_3)C_1 \\ &\quad = (\mathbf{A}\mathbf{C})B_1 - (\mathbf{A}\mathbf{B})C_1, \quad \text{by (2)}; \end{aligned}$$

similarly, the second and the third components of V**AVBC** are

$$(\mathbf{A}\mathbf{C})B_2 - (\mathbf{A}\mathbf{B})C_2,$$
$$(\mathbf{A}\mathbf{C})B_3 - (\mathbf{A}\mathbf{B})C_3,$$

respectively; hence, compounding the three components, *i.e.* adding them, after the first has been multiplied by **i**, the second by **j**, the third by **k**,

$$\mathbf{V}\mathbf{A}\mathbf{V}\mathbf{B}\mathbf{C} = (\mathbf{A}\mathbf{C})\mathbf{B} - (\mathbf{A}\mathbf{B})\mathbf{C}, \qquad (4)$$

which is, in fact, the form (4'), foreseen at the beginning, with $y = \mathbf{AC}$ and $z = -\mathbf{AB}$.*

The formula (4) is very important in practical applications.

*The formula (4) can also be proved without splitting the factors **A**, **B**, **C** into their rectangular components. See 'Problems and Exercises,' where some necessary hints are given.

Observe that V**AVBC** *does not* retain its value on cyclic permutation of its factors (as **AVBC** did); we have, indeed, according to (4),

$$\mathbf{V}\mathbf{B}\mathbf{V}\mathbf{C}\mathbf{A} = (\mathbf{B}\mathbf{A})\mathbf{C} - (\mathbf{B}\mathbf{C})\mathbf{A}, \qquad (4a)$$
$$\mathbf{V}\mathbf{C}\mathbf{V}\mathbf{A}\mathbf{B} = (\mathbf{C}\mathbf{B})\mathbf{A} - (\mathbf{C}\mathbf{A})\mathbf{B}; \qquad (4b)$$

now these are certainly three totally different vectors, since (4) lies in the plane **B**, **C**, whereas (4*a*), (4*b*) lie in the planes **C**, **A** and **A**, **B** respectively.

But it is interesting to remark that *the sum of these three vectors vanishes*; in fact the sum of the right sides of (4), (4*a*), (4*b*) vanishes identically, since $\mathbf{CA} = \mathbf{AC}$, etc; hence

$$\mathbf{V}\mathbf{A}\mathbf{V}\mathbf{B}\mathbf{C} + \mathbf{V}\mathbf{B}\mathbf{V}\mathbf{C}\mathbf{A} + \mathbf{V}\mathbf{C}\mathbf{V}\mathbf{A}\mathbf{B} = 0.$$

The particular case of (4), in which $\mathbf{C} = \mathbf{A}$, is most often met with; we have then

$$\mathbf{V}\mathbf{A}\mathbf{V}\mathbf{B}\mathbf{A} = A^2\mathbf{B} - (\mathbf{A}\mathbf{B})\mathbf{A}, \qquad (5)$$

and especially, if **A** be a *unit-vector*, say $= \mathbf{n}$, then

$$\mathbf{V}\mathbf{n}\mathbf{V}\mathbf{B}\mathbf{n} = \mathbf{B} - (\mathbf{B}\mathbf{n})\mathbf{n}. \qquad (5a)$$

Now **Bn** is the (scalar) *component* of **B** taken along **n**, and therefore (**Bn**)**n** is the **part*** of **B** along **n**, as regards both intensity and direction; subtracting it from the whole vector **B** we get its part normal to **n**. Thus we see that

$$\mathbf{V}\mathbf{n}\mathbf{V}\mathbf{B}\mathbf{n}$$

gives the part of **B** *normal to* **n**, as regards both size and direction.

If, for instance, **n** be the normal of a surface, at a given point, then V**n**V**Bn** gives immediately the part of **B** *tangent* to the surface.

In what has now been given we have all that is required for easily following the vector-algebra of the subsequent chapters on Mechanics.

There would remain the so-called '*linear vector-operator*,' which might be treated in this preparatory survey of vector-algebra. But this operator is considered in the text itself, as the 'symmetrical' operator in the chapter on Rigid Dynamics, and as the 'non-symmetrical' or general linear vector-operator in the chapter devoted to Non-rigid Bodies.

All that is still required are a few notions of elementary Vector-Analysis, to which we shall now pass.

*The '*component*' is a *scalar*, the '**part**' of a vector is a *vector*.

Differential and Integral Properties of Vectors.

Let us consider a vector **A** as a function of some independent variable scalar; to fix the ideas, let this independent variable be the time t, reckoned from some 'initial' instant.

Then, in general, both the direction and the tensor of **A** will vary with t.

To illustrate, let $\mathbf{A}=\mathbf{r}$ be the vector drawn from a fixed point O to a material particle P moving about in space along any curved path (Fig. 13); then **r** will vary with the time or **r** will be a function of t.

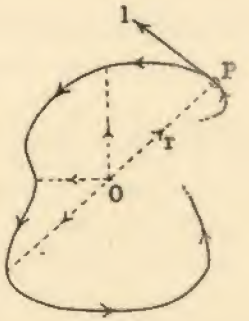

Fig. 13.

Again, in an electromagnetic field the electric and magnetic forces **E**, **M** are generally variable in time, as regards both their intensities and their directions, *i.e.* the vectors **E**, **M** are functions of t. If these functions be periodic, we have the special case of electromagnetic oscillations, the end-points of the corresponding vectors describing closed paths.

A is said to be a *continuous* vector function of the time t, if both *its direction and its tensor* vary with t in a *continuous* manner. Thus, any vector being the incarnation of three scalars, the continuity of a vector implies the continuity of three scalar functions as considered in common analysis. But here again we shall avoid any artificial decomposition, and treat the vector function **A** as a whole.

The *differentiation* of **A** with respect to t is quite as simple as the differentiation of an ordinary or scalar function. Hence, on this point, only a few remarks are needed.

To obtain an instructive picture of such differentiation we may proceed as follows.

First of all, any parallel shifting of the whole vector being, by Def. I., an indifferent matter, let us suppose that the *origin* of the changing vector **A** is fixed, say in O. Then the statement that **A** is a function of the time t means (as above for **r**) exactly the same thing as saying that the end-point of the vector **A** moves about in space; and the continuity of the vector function **A** implies the continuity of this point's motion. Let P (Fig. 14) be the

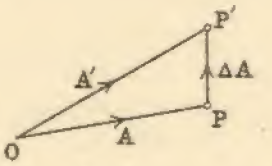

Fig. 14.

position of this point at the instant t, and P' its position at any later instant $t+\Delta t$, where Δt is a finite increment of t. Then

$$\overrightarrow{OP}=\mathbf{A}$$

is the vector-value at the instant t and

$$\overrightarrow{OP'}=\mathbf{A}'$$

at the instant $t+\Delta t$. The vector running from P to P' or

$$\overrightarrow{PP'}=\Delta\mathbf{A}=\mathbf{A}'-\mathbf{A}$$

will be the *increment* of the vector under consideration during the time Δt. The ratio

$$\frac{\Delta\mathbf{A}}{\Delta t}$$

will be the average rate of change of the vector **A** or the *mean velocity* of its end-point, in the time-interval Δt. Now, if this ratio tends to a definite limiting vector, when Δt is infinitely reduced, this limiting vector is called the *differential coefficient* of **A** with respect to t or the *flux* of **A**, and is denoted by $\frac{d\mathbf{A}}{dt}$ or by $\dot{\mathbf{A}}$; thus

$$\dot{\mathbf{A}}=\frac{d\mathbf{A}}{dt}=\operatorname{Lim}\frac{\mathbf{A}'-\mathbf{A}}{\Delta t}=\operatorname{Lim}\frac{\Delta\mathbf{A}}{\Delta t}.$$

In the present case, t being the time, $\dot{\mathbf{A}}$ is what is called the *instantaneous velocity* of the moving end-point P, as regards both absolute value and direction. If the supposed definite limit exists, $\dot{\mathbf{A}}$ is tangent to the path of P.

Thus, if P coincides with the material particle of Fig. 13,

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt}$$

will be the *velocity of the particle* along its path, at any instant t.

Now, it is seen immediately that, if

$$\mathbf{R} = \mathbf{A} + \mathbf{B},$$

$\mathbf{A}$, $\mathbf{B}$ being two vector functions of t, then

$$\dot{\mathbf{R}} = \dot{\mathbf{A}} + \dot{\mathbf{B}},$$

and similarly for a sum of three and more vector functions.

Again, if $\mathbf{a}$ be the unit of $\mathbf{A}$, or $\mathbf{A} = A\mathbf{a}$, then

$$\dot{\mathbf{A}} = \dot{A}\mathbf{a} + A\dot{\mathbf{a}};$$

thus $\dot{\mathbf{A}}$, the flux of $\mathbf{A}$, has, in general, a certain component along $\mathbf{a}$, *i.e.* along the direction of the original vector $\mathbf{A}$, and also some component along $\dot{\mathbf{a}}$, the direction of which differs from that of $\mathbf{a}$. Observe that, $\mathbf{a}$ being a unit vector, $\dot{\mathbf{a}}$ will not necessarily be so.

If the *direction* of $\mathbf{A}$ be *invariable*, then $\dot{\mathbf{a}} = 0$, and consequently

$$\dot{\mathbf{A}} = \dot{A}\mathbf{a}.$$

On the other hand, if the *tensor* of $\mathbf{A}$ be constant, then

$$\dot{\mathbf{A}} = A\dot{\mathbf{a}}.$$

In the last case the representative point P moves on the surface of a sphere of radius A.

More generally, if n be any scalar function of t, and

$$\mathbf{R} = n\mathbf{A},$$

then

$$\dot{\mathbf{R}} = \dot{n}\mathbf{A} + n\dot{\mathbf{A}}.$$

Again, considering the scalar product of a pair of variable vectors $\mathbf{A}$, $\mathbf{B}$, we have

$$\frac{d}{dt}(\mathbf{AB}) = \operatorname{Lim} \frac{(\mathbf{A}+\Delta\mathbf{A})(\mathbf{B}+\Delta\mathbf{B}) - \mathbf{AB}}{\Delta t}$$

$$= \operatorname{Lim} \left\{ \mathbf{A}\frac{\Delta\mathbf{B}}{\Delta t} + \mathbf{B}\frac{\Delta\mathbf{A}}{\Delta t} + \frac{\Delta\mathbf{A}\,.\,\Delta\mathbf{B}}{\Delta t} \right\}, \text{ by Theor. II.,}$$

whence

$$\frac{d}{dt}(\mathbf{AB}) = \mathbf{A}\frac{d\mathbf{B}}{dt} + \mathbf{B}\frac{d\mathbf{A}}{dt} = \mathbf{A}\dot{\mathbf{B}} + \mathbf{B}\dot{\mathbf{A}}, \tag{6}$$

just as for the product of a pair of ordinary scalar functions. This property follows obviously from the *distributivity* of the scalar multiplication as enunciated in Theorem II. But, by Theorem IV.,

vector multiplication is also distributive; thus, we shall get, in exactly the same manner,

$$\frac{d}{dt}\mathbf{VAB} = \mathbf{VA}\frac{d\mathbf{B}}{dt} + \mathbf{V}\frac{d\mathbf{A}}{dt}\mathbf{B} = \mathbf{VA}\dot{\mathbf{B}} + \mathbf{V}\dot{\mathbf{A}}\mathbf{B} \tag{7}$$

$$= \mathbf{VA}\dot{\mathbf{B}} - \mathbf{VB}\dot{\mathbf{A}}.$$

Here, of course, the proper order of the factors must be preserved.

Without any difficulty, the reader may prove also that

$$\frac{d}{dt}(\mathbf{AVBC}) = \dot{\mathbf{A}}\mathbf{VBC} + \mathbf{AV}\dot{\mathbf{B}}\mathbf{C} + \mathbf{AVB}\dot{\mathbf{C}}, \tag{8}$$

and similarly $\frac{d}{dt}(\mathbf{VAVBC}) = \mathbf{V}\dot{\mathbf{A}}\mathbf{VBC} + \mathbf{VAV}\dot{\mathbf{B}}\mathbf{C} + \mathbf{VAVB}\dot{\mathbf{C}}$,

or, by (4), $\frac{d}{dt}(\mathbf{VAVBC}) = (\mathbf{AC})\dot{\mathbf{B}} - (\mathbf{AB})\dot{\mathbf{C}} + (\dot{\mathbf{A}}\mathbf{C} + \mathbf{A}\dot{\mathbf{C}})\mathbf{B}$

$$- (\dot{\mathbf{A}}\mathbf{B} + \mathbf{A}\dot{\mathbf{B}})\mathbf{C}.$$

Differentials are used precisely as in ordinary analysis; thus

$$d\mathbf{A} = \frac{d\mathbf{A}}{dt}\,dt = \dot{\mathbf{A}}\,dt.$$

Second and higher differential coefficients do not require any supplementary explanation, since

$$\ddot{\mathbf{A}} \quad \text{or} \quad \frac{d^2\mathbf{A}}{dt^2}$$

may be considered simply as the flux of $\dot{\mathbf{A}}$; and so on.

Thus, returning to the example of a moving particle, the vector

$$\mathbf{w} = \frac{d\mathbf{v}}{dt} = \dot{\mathbf{v}},$$

i.e.

$$\mathbf{w} = \ddot{\mathbf{r}} = \frac{d^2\mathbf{r}}{dt^2}$$

will be the resultant or total *acceleration* of the particle. This may be easily decomposed into its tangential and normal components. For, let $\mathbf{l}$ be a unit-vector tangent to the path and concurrent with the motion, and ds an element of the path, *i.e.* the absolute value of the infinitesimal vector $d\mathbf{r}$; then $v = ds/dt$ and $\mathbf{v} = v\mathbf{l}$, whence, by differentiation,

$$\mathbf{w} = \dot{v}\mathbf{l} + v\frac{d\mathbf{l}}{dt} = \dot{v}\mathbf{l} + v\frac{ds}{dt}\frac{d\mathbf{l}}{ds},$$

or

$$\mathbf{w} = \dot{v}\mathbf{l} + v^2\frac{d\mathbf{l}}{ds}.$$

Now, $\mathbf{l}$ being a unit-vector, or $\mathbf{l}^2 = 1 = \text{const.}$, we have

$$\mathbf{l}\frac{d\mathbf{l}}{ds} = 0;$$

hence, the vector $d\mathbf{l}/ds$ is perpendicular to $\mathbf{l}$, *i.e.* *normal* to the path; besides, it is contained in the plane of the two tangents $\mathbf{l}$ and $\mathbf{l}+d\mathbf{l}$, that is in the *osculating plane* of the path. Thus the vector $d\mathbf{l}/ds$ points from the moving particle towards the centre of curvature; moreover, its tensor is easily seen to be equal to the curvature or to the inverse radius (R) of curvature at the given point of the path. Hence, denoting by $\mathbf{n}$ a unit-vector pointing from the particle towards the centre of curvature, the last equation may be written

$$\mathbf{w} = \dot{v}\mathbf{l} + \frac{v^2}{R}\mathbf{n},$$

showing that the resultant acceleration (for any path, plane or tortuous) consists of the *tangential* component

$$\dot{v} = dv/dt = d^2s/dt^2$$

and of the *normal* component

$$v^2/R,$$

which is *towards the centre of curvature*,—the well-known result of kinematics.

Here is another kinematical example, affording a beautiful illustration of the use of vector products, of their distributive property and the consequent formula (7) of their differentiation. Let a material particle P (Fig. 15) move along a *plane path* so that

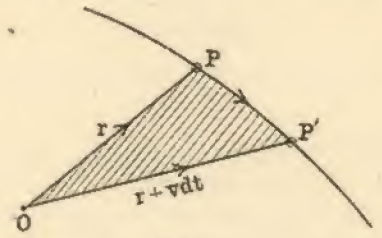

Fig. 15.

the *areas swept out by the radius vector* $\mathbf{r}$, drawn towards it from a fixed centre O in the plane of the path, *are proportional to the time*, *i.e.* so that the area swept out per unit time is *constant*,—which, for P = planet and O = sun, is Kepler's second law. Let P be the position of the particle at the instant t, and P' at the instant $t+dt$.

Then, by the definition of vector product, the area POP' swept out during dt is given by the tensor of the vector

$$dt \,.\, \mathbf{F} = \tfrac{1}{2}\mathrm{V}\mathbf{r}(\mathbf{r} + \mathbf{v}\,dt) = \tfrac{1}{2}dt \,.\, \mathrm{V}\mathbf{r}\mathbf{v},$$

since $\mathrm{V}\mathbf{r}\mathbf{r} = 0$, or per unit time,

$$\mathbf{F} = \tfrac{1}{2}\mathrm{V}\mathbf{r}\mathbf{v}.$$

Now, by the supposition, the tensor of $\mathbf{F}$ is constant, and as $\mathbf{F}$ is normal to $\mathbf{r}$, $\mathbf{v}$ and as the whole path is supposed to be in one plane, the direction of $\mathbf{F}$ is also constant, *i.e.* the whole vector $\mathbf{F}$ is invariable in time; it is an *invariant* of the system. Hence $d\mathbf{F}/dt = 0$, or by (7), $\quad \mathrm{V}\mathbf{r}\dot{\mathbf{v}} + \mathrm{V}\dot{\mathbf{r}}\mathbf{v} = 0.$

But $\dot{\mathbf{r}} = \mathbf{v}$ and $\mathrm{V}\mathbf{v}\mathbf{v} = 0$; thus, $\dot{\mathbf{v}} = \mathbf{w}$ being the acceleration,

$$\mathrm{V}\mathbf{r}\mathbf{w} = 0,$$

that is to say: *the acceleration is always towards, or from, the centre*, or the motion is *central*.

Vice versa, if the motion is supposed to be central and plane, we have $\quad \mathrm{V}\mathbf{r}\mathbf{w} = \mathrm{V}\mathbf{r}\dot{\mathbf{v}} = 0,$

and, as $\mathrm{V}\dot{\mathbf{r}}\mathbf{v}$ vanishes identically, also

$$\mathrm{V}\mathbf{r}\dot{\mathbf{v}} + \mathrm{V}\dot{\mathbf{r}}\mathbf{v} = 0$$

or $\quad \mathrm{V}\mathbf{r}\mathbf{v} = \text{const.},$

i.e. Kepler's second law.

Thus, the reciprocal equivalency of central motion and of Kepler's second law is seen by the aid of the vector language almost immediately.

If a vector, $\mathbf{A}$, be a function of two or more independent scalar variables r, s, ..., instead of d the symbol ∂ of *partial* differentiation will be used; thus

$$d\mathbf{A} = \frac{\partial \mathbf{A}}{\partial r}\,dr + \frac{\partial \mathbf{A}}{\partial s}\,ds + \dots.$$

Of particular importance, especially for physical applications, is the case of a vector, say $\mathbf{R}$, depending on three scalar variables defining the position of a point in three-dimensional space, that is to say, the case in which the vector $\mathbf{R}$ is to be considered with regard to its *distribution in space*.

It is particularly in *this* branch of vector analysis that some special concepts have been created, remarkably adapted to the very nature of vectors, concepts that are of great theoretical interest and capable of numerous applications, and which therefore require, and deserve, a special treatment.

None the less I shall not pretend to develop here fully this branch of vector analysis, but only such parts of it as may be needed for the subsequent work, and which,* in fact, will prove to be sufficient for almost any physico-mathematical purpose.

To every point of space, or of a certain portion of space, let there correspond a given vector **R**, of definite direction and tensor, generally varying from point to point. Then the space, or its portion in question, considered as the seat of these different **R**'s, is called a **vector field** or the field of **R**.

Thus, if **R** be the electric force or the magnetic force, we have an electric or magnetic field or a field of electric or magnetic force, respectively. Again, a portion of space occupied by a moving fluid will also be a vector field, namely the field of a vector representing at each point the absolute value of the velocity and the direction of the motion of the fluid.

If **R**, as regards both its tensor and direction, be constant everywhere, we have a *homogeneous* field, otherwise—a *heterogeneous* field.

Thus, the electrostatic field between the plates of a condenser (plane and parallel) is approximately homogeneous, provided that we do not approach the edges of the plates. But the vector fields met with in nature are generally heterogeneous.

A vector field is said to be *continuous*, if the corresponding vector **R**, both in direction and absolute value, varies in a continuous manner from point to point.

We shall suppose, as a rule, that the field under consideration is not only continuous but also that the vector **R** admits everywhere definite differential coefficients, at least of the first and second orders, with respect to space, *i.e.* that, if l be the (scalar) length measured in any direction, both $\frac{\partial \mathbf{R}}{\partial l}$ and $\frac{\partial^2 \mathbf{R}}{\partial l^2}$ exist as certain definite vectors.

Nearly all that is needed for the investigation of the most characteristic differential properties of a vector field is concentrated in a certain differential operator, called **the Hamiltonian** (or sometimes 'Nabla' or 'Atled') and denoted by ∇.

The best and most natural way of arriving at this differential operator, and of grasping its true meaning, is to consider, at the

* Especially if combined with the supplementary notions of vector method developed occasionally, in the subsequent chapters themselves.

starting point, certain *integral*, not differential, concepts, namely the so-called **line-integral** and the **surface-integral** of the vector **R**.

To the definition of these fundamental concepts of vector analysis we therefore now pass.

Let s (Fig. 16) be any continuous line joining any two points

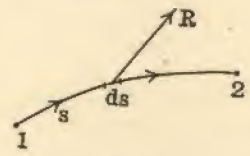

Fig. 16.

1, 2 of the vector field **R**; call $1 \to 2$ the *positive* direction of the line s. Let the infinitesimal vector $d\mathbf{s}$ represent an element of the line s, both in length and direction, the direction of $d\mathbf{s}$ being coincident with the positive direction of s at the place where the element lies.

Take the projection of **R** upon $d\mathbf{s}$ and multiply it by the (scalar) length ds of $d\mathbf{s}$, that is to say, take the scalar product

$$\mathbf{R}\, d\mathbf{s}$$

of any element $d\mathbf{s}$ of the path s and of the corresponding **R**, and sum up or integrate from 1 to 2. Then the integral

$$I_{12} = \int_1^2 \mathbf{R}\, d\mathbf{s},$$

supposing that it exists as a definite limit of a sum, is called the **line-integral** *of* **R** *taken along* s.

The line-integral, thus defined, is a *scalar*, of course. If the line of integration s be *closed*, or what is called a **circuit**, then we shall denote the line integral by

$$I_{(s)} = \int_{(s)} \mathbf{R}\, d\mathbf{s},$$

the positive sense of the circuit s being fixed in a definite manner.

Again, let σ be a continuous *surface* drawn in the vector field **R**; let us call one of its sides the *positive* (or +) *side* and the other the *negative* (or −) *side*. Let the unit vector **n** represent

the *normal* of any (scalar) element $d\sigma$ of the surface σ, crossing it from the − to the + side (see Fig. 17). The scalar product $\mathbf{Rn}$

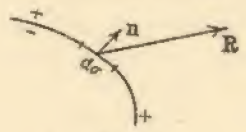

Fig. 17.

is the normal component of $\mathbf{R}$, at any point of the surface. Multiply this product by the infinitesimal area $d\sigma$ and sum up or integrate over the whole surface σ; then

$$S_\sigma = \int \mathbf{Rn}\, d\sigma,$$

if it exists, is called the **surface-integral** *of* $\mathbf{R}$ *extended over* σ.

This integral is, of course, again a scalar. In considering its value, we must always keep in mind how the positive sense of the normal $\mathbf{n}$ has been fixed. Especially, if the surface σ be a *closed* surface, it is usual to consider its outer side as positive, *i.e.* to draw $\mathbf{n}$ *outwards*.

In mathematical physics both the *line-integral* and the *surface-integral* of a vector are of frequent occurrence; this is the reason that these concepts have been constructed at all and are studied in vector analysis.

Thus, referring always to Fig. 16, if $\mathbf{F}$ is a mechanical force and s the path of the point of its application, then the line-integral

$$\int \mathbf{F}\, d\mathbf{s}$$

is the mechanical *work* done by this force.

Again, the so-called '*electromotive force*' along any line s joining a pair of arbitrary points of an electric field is the line-integral of the electric force $\mathbf{E}$,

$$\text{E.M.F.} = \int_1^2 \mathbf{E}\, d\mathbf{s}.$$

An analogous magnitude M.M.F., also of frequent use, is obtained by taking the magnetic force $\mathbf{M}$ instead of the electric $\mathbf{E}$. Of course the E.M.F. and M.M.F. depend, generally, not only on the position of the terminals 1, 2, but also on the choice of the path s leading from the first to the second.

Especially useful for the description of electromagnetic laws is

the consideration of E.M.F. and M.M.F. for *closed* paths or circuits, *i.e.* according to our notation:

$$\int_{(s)} \mathbf{E}\, d\mathbf{s} \quad \text{and} \quad \int_{(s)} \mathbf{M}\, d\mathbf{s}.$$

Again, let $\mathbf{v}$ be the velocity of fluid motion, at any point of the space occupied by a fluid, at the instant t of time; then it is often useful to consider the line-integral

$$\int_{(s)} \mathbf{v}\, d\mathbf{s},$$

called the **circulation** round the curve s. It is closely related to vortex motion, and also to the most characteristic properties of irrotational motion, especially if the fluid occupies a cyclic space, as will be seen in the chapter devoted to Hydrodynamics (Chap. VI.).

Fluid motion affords also the best illustration for the *surface-integral*. Thus, let ρ be the density of the fluid, *i.e.* its mass per unit volume, which generally may vary with time and in space, and let $\mathbf{v}$ have the above meaning. Then $\rho\mathbf{v}$ will be the current, *i.e.* the amount (mass) of fluid crossing unit area of a surface perpendicular to $\mathbf{v}$, per unit time. Hence

$$\rho v \cos(\mathbf{v}, \mathbf{n})\, d\sigma \quad \text{or} \quad \rho \mathbf{vn}\, d\sigma$$

will be the normal component of the current crossing any surface-element $d\sigma$, and the surface-integral

$$S_\sigma = \int \rho \mathbf{vn}\, d\sigma$$

will be the total amount of fluid crossing the surface σ from its negative to its positive side, or *the total current through* σ. In particular, if σ be a closed surface, then S_σ will be the amount of fluid leaving* the space limited by σ, supposing of course that fluid (mass) is neither created nor annihilated on its path.

Keeping in mind the above definitions of the line- and surface-integrals, the reader will see almost immediately the truth of the following propositions.

First of all, on inverting the sense of integration, we change only the sign of the line-integral, *i.e.*

$$I_{21} = -I_{12}, \tag{9}$$

* If $S_\sigma > 0$, and entering into, if $S_\sigma < 0$.

and, in particular, for a circuit:

$$I_{(-s)} = -I_{(s)},$$

where $-s$ denotes the opposite of s.

Secondly, if *acb* is any continuous line joining a pair of points a, b of a circuit $s = adbea$ (Fig. 18), then

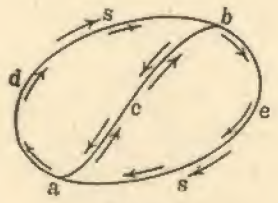

FIG. 18.

$$I_{adbca} = I_{adb} + I_{bca},$$

$$I_{acbea} = I_{bca} + I_{acb},$$

and as, by the above, $I_{bca} = -I_{acb}$, it follows:

$$I_{adb} + I_{bca} = I_{(s)} = I_{adbca} + I_{acbea},$$

or, denoting the circuit *adbca* by s_1 and the circuit *acbea* by s_2,

$$I_{(s)} = I_{(s_1)} + I_{(s_2)}.$$

Similarly, decomposing the given circuit s, by the introduction of an appropriate network of lines, into three or more circuits s_1, s_2, s_3, etc. (Fig. 19), we get

$$I_{(s)} = I_{(s_1)} + I_{(s_2)} + I_{(s_3)} + \dots . \qquad (10)$$

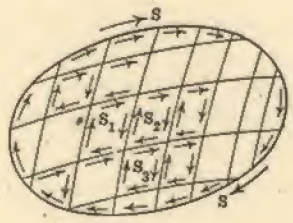

FIG. 19.

Now, consider any continuous surface σ bounded entirely by the given circuit s, and the whole network of lines drawn on this surface, as above. Take one of the sides of σ as the + side, and the other as the − side, and draw everywhere its normal $\mathfrak{n}$ crossing σ

from the − side to the + side. Let, by convention, the *positive sense of any circuit* drawn on σ be that which to an observer looking along $\mathfrak{n}$ appears *clockwise*, and consequently the negative sense that which to the same observer appears counter-clockwise. Then equation (10) may be put in words as follows:

The line-integral of $\mathfrak{R}$ taken round the perimeter of the surface σ is equal to the algebraic sum of the line-integrals of $\mathfrak{R}$ taken round the perimeters of each of the partial surfaces σ_1, σ_2, σ_3, etc., into which the whole surface σ may be split by any network of lines, all integrals being taken in the same sense (*i.e.* either all in the positive or all in the negative sense).

Now, the process of subdivision of the whole surface σ into smaller surfaces σ_1, σ_2, etc., or generally $\Delta\sigma$, may be carried on to any extent. Hence, passing to the limit, denoting by s_i the circumference of any element $\Delta\sigma$ of the surface σ and supposing that the ratio

$$\frac{I_{(s_i)}}{\Delta\sigma}$$

tends to a definite limiting value ψ, when $\Delta\sigma$ is infinitely reduced, we may conclude from the above that the *line-integral* $I_{(s)}$ must be reducible to an integral taken over the *surface* σ bounded by the circuit s:

$$\int_{(s)} \mathfrak{R}\, d\mathfrak{s} = \int \psi\, d\sigma. \qquad (11)$$

Here, by supposition, ψ is a definite (scalar) magnitude depending only on the properties of the field $\mathfrak{R}$ at the place where $d\sigma$ is taken and on the orientation of this surface-element, or on the direction of its normal $\mathfrak{n}$, but independent of the form of the circumference s_i of that $\Delta\sigma$ of which $d\sigma$ is the limit, and, generally, independent of the way by which the limit has been approached.

Now, ψ being a scalar, we may consider it as the projection of a certain vector $\mathfrak{C}$ upon the normal $\mathfrak{n}$, *i.e.* as the *normal component* of $\mathfrak{C}$:

$$\psi = \mathfrak{C}\mathfrak{n}.$$

Then (11) will assume the form

$$\int_{(s)} \mathfrak{R}\, d\mathfrak{s} = \int \mathfrak{C}\mathfrak{n}\, d\sigma\,; \qquad (12)$$

i.e. the *line-integral* of $\mathfrak{R}$ taken round s will be equal to the *surface-integral* of $\mathfrak{C}$ taken over σ, the surface σ being bounded by the circuit s.

The vector $\mathfrak{C}$ which, as we shall see a little later more explicitly,

depends only on the local properties of the vector-field **R**, *i.e.* on the distribution of **R** at the given place in the field, is called the *rotation** or the **curl** of **R**. We shall, following the example of most English writers, choose the latter name, thus writing

$$\mathbf{C} = \text{curl } \mathbf{R}.$$

Then equation (12) may be stated as follows:

Theorem V. *The line-integral of a vector* **R** *taken round the circuit s is equal to the surface-integral of its curl taken over any surface bounded by s*:

$$\int_{(s)} \mathbf{R}\, d\mathbf{s} = \int \mathbf{n} \operatorname{curl} \mathbf{R}\, d\sigma, \qquad (A)$$

the sense of the circuit in regard to the surface-normal **n** being *positive*, as explained above.

This theorem, or rather its Cartesian form, is the widely known **Theorem of Stokes.**

For the Cartesian expansion of (A), and of nearly all vector formulae of this, and also of the five following chapters, see 'Appendix,' devoted especially to such equivalences.

In the above, the component of the vector $\mathbf{C} = \text{curl } \mathbf{R}$ along any direction **n** has been defined, namely by

$$\mathbf{Cn} = \operatorname{Lim} \frac{I_{(s)}}{\Delta\sigma},$$

where $\Delta\sigma$ is normal to **n**. This is as much as to say that **Cn** is the line-integral of **R**, per unit area, taken round the circumference of an infinitesimal area, whose positive normal is **n**. Now, to obtain the resultant vector **C** itself, it is (necessary and) sufficient to determine its components along any three non-coplanar directions. As such directions, let us take, for example, the old **i**, **j**, **k**, and let the co-ordinates of any point of space measured along these axes be x, y, z, respectively. Then, going back to the above definition of **Cn** and taking in turn **n** coincident with **i** or **j** or **k**, we shall get

$$\mathbf{Ci} = C_1, \quad \mathbf{Cj} = C_2, \quad \mathbf{Ck} = C_3.$$

To obtain C_1, we have to take an infinitesimal area in the plane passing through the given point x, y, z parallel to the plane y, z, for

* Because, **R** being, for example, the (infinitesimal) displacement, in a deformable medium, $\frac{1}{2}\mathbf{C}$ is the corresponding *rotation*. See Chap. V., and also the end of Chap. IV.

example the rectangle 12341, of which x, y, z is the centre and of which dy, dz are the sides (Fig. 20). Then, remembering that in

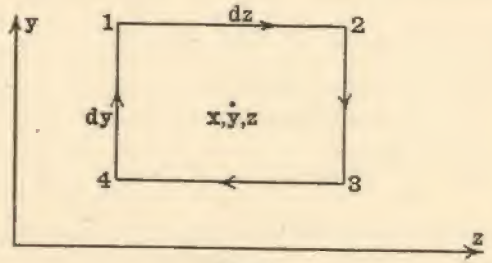

FIG. 20.

this case the proper order of integration is 12341, as indicated by arrows,* we have

$$\begin{aligned} I_{(s)} = I_{12341} &= \left(R_3 + \frac{1}{2}\frac{\partial R_3}{\partial y}dy\right)dz - \left(R_2 + \frac{1}{2}\frac{\partial R_2}{\partial z}dz\right)dy \\ &\quad - \left(R_3 - \frac{1}{2}\frac{\partial R_3}{\partial y}dy\right)dz + \left(R_2 - \frac{1}{2}\frac{\partial R_2}{\partial z}dz\right)dy \\ &= \left(\frac{\partial R_3}{\partial y} - \frac{\partial R_2}{\partial z}\right)dy\,dz, \end{aligned}$$

where (R_1), R_2, R_3 are the rectangular components of **R** at the point x, y, z. But

$$dy\,dz = d\sigma$$

is the area of the rectangle; hence the line-integral per unit area, or C_1, will be

$$C_1 = \frac{\partial R_3}{\partial y} - \frac{\partial R_2}{\partial z}.$$

Similarly, by cyclic permutation,

$$C_2 = \frac{\partial R_1}{\partial z} - \frac{\partial R_3}{\partial x}, \quad C_3 = \frac{\partial R_2}{\partial x} - \frac{\partial R_1}{\partial y}.$$

Hence the expansion of **C** or curl **R**:

$$\operatorname{curl} \mathbf{R} = \mathbf{i}\left(\frac{\partial R_3}{\partial y} - \frac{\partial R_2}{\partial z}\right) + \mathbf{j}\left(\frac{\partial R_1}{\partial z} - \frac{\partial R_3}{\partial x}\right) + \mathbf{k}\left(\frac{\partial R_2}{\partial x} - \frac{\partial R_1}{\partial y}\right), \qquad (13)$$

or in determinantal form, if ∇_1, ∇_2, ∇_3 be written, after Heaviside, instead of $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$, respectively,

$$\operatorname{curl} \mathbf{R} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \nabla_1 & \nabla_2 & \nabla_3 \\ R_1 & R_2 & R_3 \end{vmatrix}. \qquad (13a)$$

* Since **i** is normal to Fig. 20th's plane, drawn away *from* the reader, *i.e.* vertically *downwards*, if he reads his book on a horizontal table.

Now, introducing the symbol

$$\nabla = \mathbf{i}\nabla_1 + \mathbf{j}\nabla_2 + \mathbf{k}\nabla_3 = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}, \tag{14}$$

and comparing the structure of (13*a*) with that of the vector product (1*a*), we see that (13*a*) may be written

$$\operatorname{curl}\mathbf{R} = \mathrm{V}\nabla\mathbf{R}. \tag{13b}$$

As ∇_1, ∇_2, ∇_3, the 'components' of ∇, are not ordinary scalar magnitudes but scalar operators, namely differentiators, ∇ is not a vector but an operator; none the less it has the character of a vector and may be applied, as above, to a real vector $\mathbf{R}$, the result of such an operation having a perfectly definite meaning.

Allegorically, then, $\operatorname{curl}\mathbf{R}$ may be called (and in fact has been called by Heaviside) the vector product of ∇ and $\mathbf{R}$, the true meaning and the real sense of this allegory being that $\operatorname{curl}\mathbf{R}$ is the result of the operation ∇, when applied vectorially to $\mathbf{R}$.

It is this operator $\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$ which is called the **Hamiltonian**.

It may be applied to a vector either vectorially, as above, or scalarly, as we shall see a little later.

By the very *definition of curl* we are certain that $\operatorname{curl}\mathbf{R}$ cannot depend on the particular choice of the system of coordinates such as x, y, z, but only on the distribution of the vector $\mathbf{R}$ in the given field. But it will, nevertheless, be an instructive exercise for the reader to introduce instead of x, y, z another rectangular system x', y', z', with the same origin (which is immaterial), but with different directions of axes, say $\mathbf{i}', \mathbf{j}', \mathbf{k}'$, and the corresponding operator

$$\nabla' = \mathbf{i}'\frac{\partial}{\partial x'} + \mathbf{j}'\frac{\partial}{\partial y'} + \mathbf{k}'\frac{\partial}{\partial z'},$$

and to prove then that $\operatorname{curl}'\mathbf{R}$ or $\mathrm{V}\nabla'\mathbf{R}$ is identically the same vector as $\operatorname{curl}\mathbf{R}$, *i.e.* $\mathrm{V}\nabla\mathbf{R}$. Using the formulae of transformation

$$x' = ax + by + cz, \text{ etc.},$$

with constant a, b, c, etc., and the well-known conditions of orthogonality, the reader will realise this in a moment.

Observe that we arrived at the *curl* by the means of the line-integral I. Now let us repeat, *mutatis mutandis*, the whole reasoning with the surface-integral S, instead of I.

As we started with I for a closed curve, so let us now take the surface-integral $S_{(\sigma)}$ for a closed surface σ:

$$S_{(\sigma)} = \int \mathbf{R}\mathbf{n}\, d\sigma.$$

Here $\mathbf{n}$ is intended to be the *outward* normal.

First of all, then, let us divide the whole inner space τ bounded by σ into two portions τ_1, τ_2 (Fig. 21), namely by any surface σ'

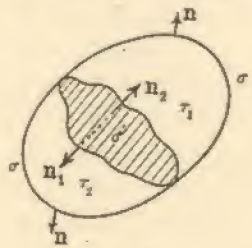

FIG. 21.

which is contained entirely in the space τ and which is bounded entirely by a curve lying on σ. Denote the whole boundary of τ_1 by σ_1 and the whole boundary of τ_2 by σ_2 and consider the sum $S_{(\sigma_1)} + S_{(\sigma_2)}$, where in the first term the normal is to be taken everywhere outwards from τ_1 and in the second outward from τ_2. Now, σ_1 consists of a part of σ and of one side of σ', and σ_2 consists of the remaining part of σ and of the other side of σ'. Thus the parts of the surface-integrals extended over both sides of σ' cancel,* and we get

$$S_{(\sigma)} = S_{(\sigma_1)} + S_{(\sigma_2)}.$$

This is exactly analogous to the above relation $I_{(s)} = I_{(s_1)} + I_{(s_2)}$.

In exactly the same way, dividing the space τ into any number of portions $\Delta\tau$ by the means of a system of surfaces, such as σ', everyone of which is a part of the boundary of two adjacent portions of the space τ, and denoting by σ_1, σ_2, etc., generally by σ_i, the complete boundaries of the particular $\Delta\tau$'s, we have

$$S_{(\sigma)} = S_{(\sigma_1)} + S_{(\sigma_2)} + \ldots = \Sigma S_{(\sigma_i)},$$

where the summation extends throughout the whole volume considered, *i.e.* throughout $\tau = \Sigma\Delta\tau$.

Now, this process of subdivision may, again, be carried on to

* Supposing that the normal components of $\mathbf{R}$ on the two sides of σ' are equal, *i.e.* supposing that σ' is not a surface of discontinuity.

any extent. Hence, passing to the limit and supposing that the ratio

$$\frac{S_{(\sigma)}}{\Delta\tau}$$

tends to a definite limiting value ρ, when $\Delta\tau$ is infinitely reduced, we get

$$S_{(\sigma)} = \int \mathbf{R}\mathbf{n}\, d\sigma = \int \rho\, d\tau.$$

The scalar ρ, thus defined, *i.e.* the limiting value of the above ratio or the limit of the surface-integral, per unit volume, is called the **divergence** of $\mathbf{R}$, and is denoted shortly by

$$\rho = \operatorname{div} \mathbf{R}.$$

Substituting this symbol, we have, analogously to Theor. V., the

Theorem VI. *The surface-integral of a vector* $\mathbf{R}$ *taken over any closed surface* σ *is equal to the space-integral of the divergence of* $\mathbf{R}$ *taken throughout the volume* τ *enclosed entirely by* σ:

$$\int \mathbf{R}\mathbf{n}\, d\sigma = \int \operatorname{div} \mathbf{R}\, .\, d\tau, \qquad (8)$$

$\mathbf{n}$ being the *outward* normal of σ.

This theorem has been proved under the supposition (see footnote, p. 37) that the field $\mathbf{R}$ is continuous throughout the whole region τ. But if this be not the case, and if, say, σ' be a surface of discontinuity dividing τ into two distinct portions τ_1 and τ_2, then, by applying the above theorem to each of these in turn and summing up, side by side, the two equations thus obtained, we get at once

$$\int \mathbf{R}\mathbf{n}\, d\sigma = \int \operatorname{div} \mathbf{R}\, .\, d\tau - \int (\mathbf{R}_- - \mathbf{R}_+)\mathbf{n}'\, d\sigma', \qquad (8')$$

where $\mathbf{R}_+$, $\mathbf{R}_-$ are the values of $\mathbf{R}$ at the positive and negative sides of σ', respectively, and $\mathbf{n}'$ is the normal of σ' crossing this surface of discontinuity from its negative to its positive side.

To obtain the expression for $\operatorname{div} \mathbf{R}$ in terms of differentiations

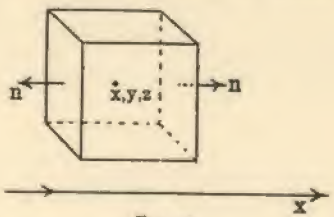

FIG. 22.

with respect to x, y, z, consider an infinitesimal parallelopiped with edges dx, dy, dz parallel to $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ respectively (Fig. 22). Then,

if x, y, z be the coordinates of the centre of the parallelopiped, the pair of sides dy, dz normal to $\mathbf{i}$ contributes to the surface-integral the sum

$$\left(R_1 + \frac{1}{2}\frac{\partial R_1}{\partial x}\, dx\right) dy\, dz - \left(R_1 - \frac{1}{2}\frac{\partial R_1}{\partial x}\, dx\right) dy\, dz = \frac{\partial R_1}{\partial x}\, dx\, dy\, dz;$$

similarly, the contributions from the two other pairs of sides of the parallelopiped, normal to $\mathbf{j}$ and $\mathbf{k}$, are, respectively,

$$\frac{\partial R_2}{\partial y}\, dy\, dz\, dx, \qquad \frac{\partial R_3}{\partial z}\, dz\, dx\, dy.$$

Hence the surface-integral will be

$$\left(\frac{\partial R_1}{\partial x} + \frac{\partial R_2}{\partial y} + \frac{\partial R_3}{\partial z}\right) dx\, dy\, dz.$$

But $dx\, dy\, dz = d\tau$ is the volume of the parallelopiped; hence the surface-integral per unit volume or, by the above definition, the *divergence* of $\mathbf{R}$ will be

$$\operatorname{div} \mathbf{R} = \frac{\partial R_1}{\partial x} + \frac{\partial R_2}{\partial y} + \frac{\partial R_3}{\partial z} = \nabla_1 R_1 + \nabla_2 R_2 + \nabla_3 R_3. \qquad (15)$$

Now, comparing this with the Cartesian expansion (2) of the scalar product of a pair of vectors, we may write shortly

$$\operatorname{div} \mathbf{R} = \nabla \mathbf{R}. \qquad (15a)$$

Both the *curl* and the *divergence* are independent of the choice of the system of coordinates and, generally, of any system of reference, and depend only on the properties of the given vector-field $\mathbf{R}$, and, as the reader will see from the subsequent chapters, both are specially characteristic of such a field.

Now, it is remarkable that both of these important magnitudes, the first a vector and the second a scalar, are obtained by the application of one and the same operator, namely the Hamiltonian. In fact, we have (13*b*) and (15*a*), which formulae tell us that *the Hamiltonian* ∇ *if applied to a vector vectorially gives its curl, and if applied scalarly gives its divergence.*

The most immediate kinematical illustration of *curl* is given at the end of Chap. IV., where it is shown explicitly that if $\mathbf{v}$ be the resultant velocity of any point of a rigid body, moving in the most general way, then *curl* $\mathbf{v}$ is twice the *angular velocity* of the body.

A larger application of *curl*, namely to deformable bodies, in which the rotation generally varies from point to point, is developed fully in Chap. V. In the same chapter frequent use will be made

also of *div*. Nevertheless we may illustrate here, also, the meaning of this operator in an immediate way, namely by our previous example of fluid motion, in which it has been shown that

$$S_{(\sigma)} = \int \rho \mathbf{v} \mathbf{n} \, d\sigma,$$

taken for any closed surface σ, gives the amount of fluid leaving the space (τ) limited by σ. Now, by the very definition of *divergence*,

$$\operatorname{div}(\rho \mathbf{v}) \, . \, d\tau$$

is the same thing as the surface-integral taken over the surface enclosing the elementary volume $d\tau$. Hence $div\,(\rho \mathbf{v})\, d\tau$ is the amount of fluid leaving the volume-element, and therefore

$$\operatorname{div}(\rho \mathbf{v})$$

the *amount leaving it, per unit volume* (and per unit time, of course), whence also the name of 'divergence.'

The meaning of the Theorem VI. and of its short formula (B) becomes now quite obvious; in fact, if we put

$$\mathbf{R} = \rho \mathbf{v},$$

it says simply that the amount crossing the surface σ is equal to the algebraic sum of all amounts leaving the elements of the volume τ enclosed entirely by this surface.

The fundamental Theorems V. and VI. bring the curl and divergence of a vector into close connection with its line-integral and surface-integral, respectively.

Combining both theorems, we may now prove an important general property of curl $\mathbf{R}$, without recurring to any system of reference.

Let σ_1, σ_2 (Fig. 23) be a pair of surfaces bounded by the same

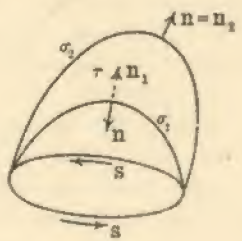

FIG. 23.

circuit s, and themselves enclosing completely a portion of space τ.

Then, by Theor. V.,

$$\int_{(s)} \mathbf{R} \, d\mathbf{s} = \int_{\sigma_1} \mathbf{n}_1 \operatorname{curl} \mathbf{R} \, d\sigma_1 = \int_{\sigma_2} \mathbf{n}_2 \operatorname{curl} \mathbf{R} \, d\sigma_2,$$

or

$$\int_{\sigma_1} \mathbf{n}_1 \operatorname{curl} \mathbf{R} \, d\sigma_1 - \int_{\sigma_2} \mathbf{n}_2 \operatorname{curl} \mathbf{R} \, d\sigma_2 = 0.$$

Hence, considering the closed surface $\sigma = \sigma_1 + \sigma_2$, denoting its outward normal by $\mathbf{n}$ and observing that $\mathbf{n}_2 = \mathbf{n}$, whereas $\mathbf{n}_1 = -\mathbf{n}$,

$$\int_{(\sigma)} \mathbf{n} \operatorname{curl} \mathbf{R} \, d\sigma = 0.$$

Now, curl $\mathbf{R}$ being a vector, we can apply to it the Theor. VI.; hence, by the last equation,

$$\int \operatorname{div} \operatorname{curl} \mathbf{R} \, . \, d\tau = 0.$$

But the above construction may be made for any circuit s, as small as we please; and consequently, reducing accordingly the closed surface σ, the last equation is seen to hold good for any portion of space τ, as small as we please. Hence

$$\operatorname{div} \operatorname{curl} \mathbf{R} = 0 \qquad (16)$$

everywhere. This may also be verified to be identically true by using the expansions (13) and (15) of *curl* and *div*, and by observing that $\partial^2 R_3 / \partial x \, \partial y = \partial^2 R_3 / \partial y \, \partial x$, etc.

It is worth noticing that (16) may also be written

$$\nabla \nabla \mathbf{R} = 0. \qquad (16a)$$

Thus, also in this respect the Hamiltonian ∇ behaves like a simple vector. Remember that, by the fundamental property of the vector product, $\mathbf{A} \times \mathbf{A} \mathbf{R} = 0$ identically.

If $\operatorname{div} \mathbf{R} = 0$ throughout the whole field, then the vector-field $\mathbf{R}$ is said to be **solenoidal** or **sourceless.***

And if curl $\mathbf{R} = 0$ everywhere, then the field $\mathbf{R}$ is called **irrotational.**†

Thus, the identity (16) may be put in words by saying that the field derived from any field $\mathbf{R}$ by 'curling' it, is always a solenoidal field, or—briefly—that *the curl of any vector is solenoidal.*

Instructive examples of solenoidal and of irrotational fields will be found in the subsequent chapters on Mechanics; and as to

* The reason of the last name is obvious by one of the above illustrations, and will be seen again in Chap. IV.

† That is *non-rotational*; remember that *curl* is closely related to rotation.

Electromagnetism, it will suffice here to observe briefly that the total electric current is always and everywhere solenoidal,* and that an electrostatic field, for example, is irrotational.

We have already admired the efficiency of the Hamiltonian ∇ in two cases, namely in getting the *curl* and the *div* of a vector. Now, to see another performance of the same operator, *i.e.* a third property of ∇, let us consider a *purely irrotational* field $\mathbf{R}$, that is such that the condition

$$\text{curl}\,\mathbf{R}=0$$

is satisfied throughout the whole space. Then, by Theor. V., if s be any closed path, or circuit,

$$I_{(s)}=\int_{(s)}\mathbf{R}\,d\mathbf{s}=0.$$

Now, taking on s any pair of points 1, 2 (Fig. 24), we have

$$0=I_{(s)}=I_{1a2}+I_{2b1}=I_{1a2}-I_{1b2};$$
$$\therefore\ I_{1a2}=I_{1b2}.$$

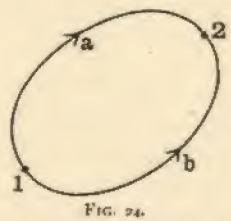

FIG. 24.

Thus, the line-integral of $\mathbf{R}$ has for the two paths a, b, and hence also for all possible continuous paths leading from 1 to 2, one and the same value. Or, in other terms, the integral

$$I_{12}=\int_1^2\mathbf{R}\,d\mathbf{s}$$

is a *single-valued function of the position of the terminal points* 1 *and* 2 in the field. Hence, if any fixed point O be chosen once and for ever as the starting point, and if p be any other point of the irrotational field $\mathbf{R}$,

$$\int_0^p\mathbf{R}\,d\mathbf{s}=\phi, \tag{17}$$

where ϕ *is a single-valued function of the position of* p *alone.*

This function ϕ is called the (scalar) **potential**† of the vector $\mathbf{R}$.

*And this is one of the most characteristic features of Maxwell's theory.

†This is the common use in general mechanics and hydromechanics, whereas in electrostatics and magnetostatics, not the above ϕ but $-\phi$, *i.e.* $\int_p^0\mathbf{R}\,d\mathbf{s}$, is called the potential. This, of course, is a matter of convention.

If the region of irrotationality of the field $\mathbf{R}$ does not occupy the whole space but only a certain portion τ of space, and if this portion of space be *cyclic* or *multiply-connected*, the vector $\mathbf{R}$ has still a potential ϕ in this region τ, but then ϕ is no longer a single-valued but generally a many-valued function of position. For further particulars regarding this subject see Chap. VI.

But, at any rate, whether ϕ be a single- or a many-valued function of position, we have, by (17), the differential property

$$\mathbf{R}\,d\mathbf{s}=d\phi=\frac{\partial\phi}{\partial s}\,ds,$$

which holds good at such places where the field is irrotational. Here ds means the tensor of $d\mathbf{s}$, so that, if R_s be the (scalar) component of $\mathbf{R}$ along $d\mathbf{s}$, we have $\mathbf{R}\,d\mathbf{s}=R_s\,ds$, and consequently

$$R_s=\frac{\partial\phi}{\partial s}. \tag{18}$$

Now, as $d\mathbf{s}$ may have any direction whatever, the formula (18) enables us to write down any component of $\mathbf{R}$, and hence also the resultant vector $\mathbf{R}$ in terms of its potential. Thus, taking for example $d\mathbf{s}=\mathbf{i}\,dx$, or $\mathbf{j}\,dy$, or $\mathbf{k}\,dz$, we have, respectively,

$$R_1=\frac{\partial\phi}{\partial x},\quad R_2=\frac{\partial\phi}{\partial y},\quad R_3=\frac{\partial\phi}{\partial z},$$

and consequently

$$\mathbf{R}=\mathbf{i}\frac{\partial\phi}{\partial x}+\mathbf{j}\frac{\partial\phi}{\partial y}+\mathbf{k}\frac{\partial\phi}{\partial z}=(\mathbf{i}\nabla_1+\mathbf{j}\nabla_2+\mathbf{k}\nabla_3)\phi$$

or

$$\mathbf{R}=\nabla\phi. \tag{19}$$

Thus, the Hamiltonian appears again. Here, applied to the potential ϕ, it gives the vector in question, $\mathbf{R}$.

Remark that ϕ is a scalar, so that $\nabla\phi$ does not require any further explanation.

Generally, $\nabla\phi$ is called the **slope** or **gradient** of the scalar function ϕ. Thus, in hydrokinematics the velocity of fluid motion is the slope of the 'velocity-potential,' if, of course, such a potential exists, *i.e.* if there are no vortices at the place considered (see Chap. V.).

Any surface $\phi=const.$ is called an *equipotential surface.*

Now, taking $d\mathbf{s}$ tangential to such a surface, we have $\partial\phi/\partial s=0$, and consequently, by the above formula,

$$\mathbf{R}\,d\mathbf{s}=0;$$

and since this is true for any tangential direction, we see that the vector $\mathbf{R}$ is *normal to the equipotential surfaces*.

At the same time it is seen that the (positive) direction of $\mathbf{R}$ is the direction of the most rapid increase of ϕ. Hence, denoting this direction of most rapid increase by the unit-vector $\mathbf{n}$ and using the common symbol $\partial/\partial n$ of partial derivation taken in the direction $\mathbf{n}$, we can write also

$$\mathbf{R} = \nabla\phi = \mathbf{n}\frac{\partial\phi}{\partial n}.$$

Thus, the Hamiltonian, in its application to a *scalar* function, might have been defined from the beginning as the slope of this function or, symbolically, as the operator

$$\nabla = \mathbf{n}\frac{\partial}{\partial n}. \tag{20}$$

Then, developing this into $\nabla = \mathbf{i}\,\partial/\partial x + \mathbf{j}\,\partial/\partial y + \mathbf{k}\,\partial/\partial z$ and applying it either scalarly or vectorially to a vector $\mathbf{R}$, we could arrive in this manner to the divergence and curl of $\mathbf{R}$. But this did not seem to me as natural a way as that chosen above; it would be essentially more artificial, and then it would not enable us to see the truth of the Theorems V. and VI. in such an immediate manner as the method adopted.

But without entering any further into similar comparisons of didactic character, let us return to the Hamiltonian itself.

Having already recognised it in its whole generality, *i.e.* in its threefold character as *slope*, *divergence* and *curl*,

$$\nabla\phi = \text{slope of } \phi, \quad \nabla\mathbf{R} = \operatorname{div}\mathbf{R}, \quad \mathrm{V}\nabla\mathbf{R} = \operatorname{curl}\mathbf{R},$$

what we need still are but a few remarks regarding the application of this marvellous operator.

First of all, the operator ∇ is distributive, since, apart from its vectorial peculiarities, it is a simple differentiator. This property holds good not only for a sum of scalars, as ϕ, ψ, but also for a sum of vectors, as $\mathbf{R}$, $\mathbf{S}$, *i.e.* not only

$$\nabla(\phi+\psi) = \nabla\phi + \nabla\psi, \tag{21}$$

but also
$$\nabla(\mathbf{R}+\mathbf{S}) = \operatorname{div}(\mathbf{R}+\mathbf{S}) = \nabla\mathbf{R} + \nabla\mathbf{S} = \operatorname{div}\mathbf{R} + \operatorname{div}\mathbf{S}, \tag{22}$$

and similarly
$$\mathrm{V}\nabla(\mathbf{R}+\mathbf{S}) = \operatorname{curl}(\mathbf{R}+\mathbf{S}) = \operatorname{curl}\mathbf{R} + \operatorname{curl}\mathbf{S}. \tag{23}$$

Then, the application of ∇ to a product of a pair of scalar functions

does not present any difficulty; since $\frac{\partial}{\partial x}(\phi\psi) = \phi\,\partial\psi/\partial x + \psi\,\partial\phi/\partial x$, etc., we have simply

$$\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi, \tag{24}$$

i.e.
$$\text{slope}\,(\phi\psi) = \phi\,\text{slope}\,\psi + \psi\,\text{slope}\,\phi.$$

Similarly, the application of ∇ to a scalar product of a pair of vectors does not bring in anything new; $\nabla(\mathbf{RS})$ is simply the slope of the scalar magnitude $\mathbf{RS}$ or of $RS\cos\theta$, if the angle included by $\mathbf{R}$, $\mathbf{S}$ is θ; thus we may write

$$\nabla(\mathbf{RS}) = RS\nabla(\cos\theta) + S\cos\theta\nabla R + R\cos\theta\nabla S.$$

But, as far as I know, it is never met with in practical applications.

Of greater importance is the application of ∇ to the vector-product of a pair of vectors, $\mathrm{V}\mathbf{RS}$. Since this is a vector we may operate on it by ∇ either scalarly or vectorially, thus giving rise to

$$\operatorname{div}\mathrm{V}\mathbf{RS} = \nabla\mathrm{V}\mathbf{RS}$$

and
$$\operatorname{curl}\mathrm{V}\mathbf{RS} = \mathrm{V}\nabla\mathrm{V}\mathbf{RS}.$$

Let us consider the first of these expressions. If ∇ were a real vector, say $\mathbf{A}$, then we should have, by Theor. III.,

$$\mathbf{A}\mathrm{V}\mathbf{RS} = \mathbf{R}\mathrm{V}\mathbf{SA} = \mathbf{S}\mathrm{V}\mathbf{AR} = -\mathbf{R}\mathrm{V}\mathbf{AS};$$

consequently, if the vector $\mathbf{S}$ were constant, in space, we should have, writing again ∇ instead of $\mathbf{A}$,

$$\nabla\mathrm{V}\mathbf{RS} = \mathbf{S}\mathrm{V}\nabla\mathbf{R} = \mathbf{S}\operatorname{curl}\mathbf{R};$$

in the same way, if $\mathbf{R}$ were constant and $\mathbf{S}$ variable, we should have
$$\nabla\mathrm{V}\mathbf{RS} = -\mathbf{R}\mathrm{V}\nabla\mathbf{S} = -\mathbf{R}\operatorname{curl}\mathbf{S}.$$

Hence, both $\mathbf{R}$ and $\mathbf{S}$ being variable, we have

$$\nabla\mathrm{V}\mathbf{RS} = \operatorname{div}\mathrm{V}\mathbf{RS} = \mathbf{S}\operatorname{curl}\mathbf{R} - \mathbf{R}\operatorname{curl}\mathbf{S}. \tag{25}$$

This formula is of particular importance in Electromagnetism; it serves, for instance, to show almost immediately that, $\mathbf{R} = \mathbf{E}$ and $\mathbf{S} = \mathbf{M}$ being the electric and the magnetic force, respectively, their vector product $\mathrm{V}\mathbf{EM}$, multiplied by the (scalar) velocity of light in vacuo, gives the flux of electromagnetic energy, per unit time and unit area, *i.e.* the so-called Poynting-vector, which already has been mentioned.

In the same way, the application of the formula (4) of Vector Algebra to the second of the above expressions gives

$$\mathrm{V}\nabla\mathrm{V}\mathbf{RS} = \mathbf{R}(\nabla\mathbf{S}) - \mathbf{S}(\nabla\mathbf{R}) + (\mathbf{S}\nabla)\mathbf{R} - (\mathbf{R}\nabla)\mathbf{S},$$

i.e.
$$\operatorname{curl}\mathrm{V}\mathbf{RS} = \mathbf{R}\,.\operatorname{div}\mathbf{S} - \mathbf{S}\,.\operatorname{div}\mathbf{R} + (\mathbf{S}\nabla)\mathbf{R} - (\mathbf{R}\nabla)\mathbf{S}, \tag{26}$$

where $(\mathbf{S}\nabla)$ is an operator composed of $\mathbf{S}$ and ∇ in the same way as the scalar product of a pair of real vectors, that is in Cartesians, for instance,

$$(\mathbf{S}\nabla) = S_1\frac{\partial}{\partial x} + S_2\frac{\partial}{\partial y} + S_3\frac{\partial}{\partial z}. \tag{27}$$

The same meaning is to be attributed to $(\mathbf{R}\nabla)$ in (26).

Having thus obtained the required formulae (25), (26) by a method which may seem dubious to the reader, it is desirable to verify the validity of these formulae, and consequently the legitimacy of the short and almost brutal method adopted. Now, this is done in a moment by Cartesian development. Thus

$$\begin{aligned}\operatorname{div} \mathrm{V}\mathbf{RS} &= \frac{\partial}{\partial x}(R_2S_3 - R_3S_2) + \frac{\partial}{\partial y}(R_3S_1 - R_1S_3) + \frac{\partial}{\partial z}(R_1S_2 - R_2S_1)\\ &= S_1\left(\frac{\partial R_3}{\partial y} - \frac{\partial R_2}{\partial z}\right) + \text{etc.} - R_1\left(\frac{\partial S_3}{\partial y} - \frac{\partial S_2}{\partial z}\right) - \text{etc.}\\ &= \mathbf{S}\operatorname{curl}\mathbf{R} - \mathbf{R}\operatorname{curl}\mathbf{S}. \quad \text{Q.E.D.}\end{aligned}$$

In the same way the reader may verify the formula (26).

In the above we had the opportunity of encountering the operator $(\mathbf{S}\nabla)$, as developed in (27). As it occurs rather often in practice, it may deserve here a few remarks. This operator is composed, scalarly, of $\mathbf{S}$, which means *any real vector*, and of the *Hamiltonian* ∇, which we may call, after the example of Oliver Heaviside, a '*fictitious vector*.'

All directional peculiarities of $\mathbf{S}$ and ∇, if considered separately, disappeared after they have been melted together into the scalar composition. That is the reason why the operator $(\mathbf{S}\nabla)$ has a purely *scalar character*. It does *not* change the nature of the magnitude operated on; that is to say, when applied to a scalar ϕ it gives a scalar, and if applied to a vector $\mathbf{R}$ it gives a vector; thus

$$(\mathbf{S}\nabla)\phi = S_1\frac{\partial\phi}{\partial x} + S_2\frac{\partial\phi}{\partial y} + S_3\frac{\partial\phi}{\partial z} = \mathbf{S}(\nabla\phi)$$

or simply $\quad = \mathbf{S}\nabla\phi = \textit{scalar},$

the parentheses in this case being superfluous; again,

$$\begin{aligned}(\mathbf{S}\nabla)\mathbf{R} &= (\mathbf{S}\nabla)(\mathbf{i}R_1 + \mathbf{j}R_2 + \mathbf{k}R_3)\\ &= \mathbf{i}(\mathbf{S}\nabla)R_1 + \mathbf{j}(\mathbf{S}\nabla)R_2 + \mathbf{k}(\mathbf{S}\nabla)R_3 = \textbf{vector}.\end{aligned}$$

Here the parentheses are necessary, since $\mathbf{S}(\nabla\mathbf{R})$ has a different meaning from $(\mathbf{S}\nabla)\mathbf{R}$, namely

$$\mathbf{S}(\nabla\mathbf{R}) = \mathbf{S}\,.\operatorname{div}\mathbf{R}.$$

Observe that $\nabla\phi$ is the slope of ϕ, and consequently $(\mathbf{S}\nabla)\phi$ or $\mathbf{S}\nabla\phi$ is the *component* of the slope of ϕ taken along $\mathbf{S}$ and *multiplied by the tensor S*. Particularly, if $\mathbf{s}$ be a unit-vector, $\mathbf{s}\nabla\phi$ is simply the *component of the slope of* ϕ *along* $\mathbf{s}$ or

$$\mathbf{s}\nabla\phi = \frac{\partial\phi}{\partial s}, \tag{28}$$

so that $\mathbf{s}\nabla$ is the symbol of what is commonly called *axial differentiation*, *i.e.* differentiation along the direction of $\mathbf{s}$. And if $\mathbf{s} = \mathbf{n}$ be the direction of the most rapid increase of ϕ, *i.e.* the direction of the resultant slope, then

$$\mathbf{n}\nabla\phi = \frac{\partial\phi}{\partial n},$$

and this follows also immediately from the previous formula (20), after scalar multiplication of both sides by $\mathbf{n}$, and remembering that $\mathbf{n}^2 = 1$.

Finally, let us consider some *iterations* or repetitions of the Hamiltonian ∇, such at least as are met with most often in the physicist's practice.

In the first place, if ϕ be any scalar function, then, as we already know, $\nabla\phi$ is a vector. Now, it may again be operated on by ∇, either vectorially or scalarly. In the *first* case, we have

$$\mathrm{V}\nabla\nabla\phi = \operatorname{curl}\nabla\phi = 0, \tag{29}$$

identically, since $\dfrac{\partial^2\phi}{\partial y\,\partial z} - \dfrac{\partial^2\phi}{\partial z\,\partial y} = 0$, etc. Thus, a vector *which has a (scalar) potential is irrotational*. Observe particularly the form $\mathrm{V}\nabla\nabla\phi = 0$, and compare it with the identity $\mathrm{V}\mathbf{AA}\phi = 0$ which occurs for any real vector $\mathbf{A}$; thus, also in this regard, ∇ behaves like a real vector.

Again, in the *second* case, we have

$$\nabla(\nabla\phi) = \operatorname{div}\nabla\phi;$$

in this case the omission of the parentheses cannot give rise to any misunderstanding, and consequently we may write also

$$\nabla\nabla\phi, \text{ or more shortly } \nabla^2\phi = \operatorname{div}\nabla\phi. \tag{30}$$

As to the Cartesian expansion, remember that, by (15),

$$\operatorname{div}\nabla\phi = \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) + \ldots = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}.$$

Again, taking the scalar square of ∇, *i.e.*

$$(\mathbf{i}\nabla_1 + \mathbf{j}\nabla_2 + \mathbf{k}\nabla_3)^2,$$

and remembering that $\mathbf{ij}=0$, etc., $\mathbf{i}^2=1$, etc., we get

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}, \tag{31}$$

that is, when applied to ϕ, precisely the same thing as above for $\operatorname{div}\nabla\phi$.

Thus, the last operator, ∇^2 which is widely known as the **Laplacian**, and which has a purely *scalar character*, may be defined simply as the *scalar iteration of the Hamiltonian* ∇. It is of very great importance in nearly all branches of mathematical physics, especially in gravitational problems, in Electromagnetism, in non-rigid Dynamics, in Conduction of Heat and in Diffusion.

Since ∇^2 is an operator of scalar character, it may also be applied to a vector $\mathbf{R}$, without any further explanation. Thus

$$\nabla^2\mathbf{R}=\nabla^2(\mathbf{i}R_1+\ldots)=\mathbf{i}\nabla^2R_1+\mathbf{j}\nabla^2R_2+\mathbf{k}\nabla^2R_3; \tag{32}$$

this is a vector as well as $\mathbf{R}$ itself. In the same way $\nabla^2\phi$ was a scalar, as was ϕ itself.

In (32), the operand being any vector $\mathbf{R}$, the operator ∇ has been taken both times scalarly. Now, what remains still to do is to apply ∇ to $\mathbf{R}$:

1° first vectorially and then scalarly, giving rise to div curl $\mathbf{R}$,

2° both times vectorially, giving rise to curl curl $\mathbf{R}$.

Remark that the third eventuality, curl div $\mathbf{R}$, is pure nonsense, since, div $\mathbf{R}$ being a scalar, its curl is meaningless.

Now, in the case 1° we have seen that

$$\operatorname{div}\operatorname{curl}\mathbf{R}=0, \text{ identically.}$$

Thus, what remains to be considered is the case 2°. Now, curl curl $\mathbf{R}$, *i.e.* the curl of the curl of $\mathbf{R}$, which is usually denoted by $\operatorname{curl}^2\mathbf{R}$, is the same thing as

$$\mathbf{V}\nabla\mathbf{V}\nabla\mathbf{R}.$$

Hence, treating ∇ as a vector and applying the formula (4), or rather its special case (5), we get

$$\operatorname{curl}^2\mathbf{R}=\nabla(\nabla\mathbf{R})-\nabla^2\mathbf{R},$$

i.e.

$$\operatorname{curl}^2\mathbf{R}=\nabla\operatorname{div}\mathbf{R}-\nabla^2\mathbf{R}, \tag{33}$$

or, in words: curl of curl = slope of divergence *minus* Laplacian.

This formula, which is also of considerable practical importance, may again be verified by Cartesian expansion.

As regards the application of the above elementary notions of Vector Method to Mechanics, it may be remarked here that the

Hamiltonian ∇ will be needed in Chapters II., III., IV. only in its simplest aspect, *i.e.* as the slope of a *scalar* function. The operators *curl* and *div* or $\mathbf{V}\nabla$ and ∇ in application to *vector* functions will not occur before Chap. V., which treats of deformable bodies.

Meanwhile, therefore, let us here put together and mark with Roman numbers the few vector formulae which will be completely sufficient for the Mechanics of a particle or of a system of material particles and, in particular, for Rigid Dynamics—Chap. II., III., IV., respectively. To avoid digressions, I shall cite them shortly, if necessary, by reference to the Roman numeral.

$$\mathbf{AB}=\mathbf{BA}. \tag{I.}$$

$$\mathbf{A}(\mathbf{B}+\mathbf{C}+\ldots)=\mathbf{AB}+\mathbf{AC}+\ldots. \tag{II.}$$

$$\mathbf{VAB}=-\mathbf{VBA}. \tag{III.}$$

$$\mathbf{VA}(\mathbf{B}+\mathbf{C}+\ldots)=\mathbf{VAB}+\mathbf{VAC}+\ldots. \tag{IV.}$$

$$\mathbf{AA}=\mathbf{A}^2=A^2. \tag{V.}$$

$$\mathbf{VAA}=0. \tag{VI.}$$

$$\mathbf{AVAB}=0, \quad \mathbf{BVAB}=0. \tag{VII.}$$

$$\mathbf{AVBC}=\mathbf{BVCA}=\mathbf{CVAB}. \tag{VIII.}$$

$$\mathbf{VAVBC}=(\mathbf{AC})\mathbf{B}-(\mathbf{AB})\mathbf{C}. \tag{IX.}$$

$$\nabla\phi=\mathbf{n}\frac{\partial\phi}{\partial n}. \tag{X.}$$

Other formulae belonging to *pure* vector calculus, *i.e.* independent of any mechanics, will be given, on the basis of the above preparatory sketch of this modern mathematical language, always under Roman numerals, (XI.) etc., in the following chapters, as the need arises in the subject under consideration.

CHAPTER II.

GENERAL PRINCIPLES.

D'Alembert's Principle.

Consider any system of material particles 1, 2, etc., of which the masses are m_1, m_2, etc., and which are acted on by what are technically termed the '*impressed*' forces $\mathbf{F}_1$, $\mathbf{F}_2$, etc., respectively.

The position of the particle 1 at the instant of time t, relative to some point of reference O, chosen at random but once and for ever, will be determined completely by the vector $\mathbf{r}_1$, of which the length $\overline{O1}$ is the tensor and $O \to 1$ the direction (Fig. 25).

Fig. 25.

Similarly will the vectors $\mathbf{r}_2$, $\mathbf{r}_3$ (generally $\mathbf{r}_i$) determine the instantaneous positions of the other particles, viz. 2, 3 (generally i). Omitting, for the sake of brevity, the index, we shall denote simply by $\mathbf{r}$ the vector corresponding to any one of these particles, and by m its mass. We could say, shortly, that, for our purposes, a material particle is characterised by one vector and one scalar (position and mass).

The instantaneous velocity of such a particle, as regards both direction and absolute value, will be given by the vector

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}},$$

and, similarly, the acceleration by the vector

$$\dot{\mathbf{v}} = \frac{d^2\mathbf{r}}{dt^2} = \ddot{\mathbf{r}}.$$

It is scarcely necessary to say that the directions of the vectors $\mathbf{r}$, $\dot{\mathbf{r}}$, $\ddot{\mathbf{r}}$ will, generally, be different from one another.

If the system be free, we should have simply for every particle separately, by Newton, $m\ddot{\mathbf{r}} = \mathbf{F}$ (*a*)

or $m\ddot{\mathbf{r}} - \mathbf{F} = 0$. But if the system is constrained, these vectors, say

$$m\ddot{\mathbf{r}} - \mathbf{F} = \mathbf{S},$$

will generally not vanish. The name of '*lost forces*' was long ago given to the vectors $-\mathbf{S}$.*

Let $\delta\mathbf{r}$ be an infinitesimal **virtual displacement** of one of the particles, *i.e.* a displacement permitted by the connections of the system. If, to give an example, the particle 1 is constrained to remain always at a constant distance from the point O, we have

$$r_1^2 = \mathbf{r}_1^2 = \text{const.},$$

so that $\mathbf{r}_1 \delta\mathbf{r}_1 = 0$;

that is to say, the virtual displacement $\delta\mathbf{r}_1$ is perpendicular to the vector $\mathbf{r}_1$ or tangential to the sphere of radius $r_1 = \text{const.}$,—a statement which after all is only a slightly changed enunciation of the original supposition. If, to take another example, two other particles 2, 3 are connected by a rigid bar of length l, we have

$$(\mathbf{r}_2 - \mathbf{r}_3)^2 = l^2 = \text{const.},$$

whence $(\mathbf{r}_2 - \mathbf{r}_3)(\delta\mathbf{r}_2 - \delta\mathbf{r}_3) = 0$,

an equation which tells simply that the difference of the displacements of the particles 2 and 3, or their *relative* displacement, must be normal to the bar.

Observe the *scalar character* of the connections in both examples given above: $r_1 = const.$, $l = const.$ Every one of such *scalar conditions* takes away *one* degree of freedom of the system. We

* For the history of this subject, see for instance: E. Mach, *Die Mechanik in ihrer Entwickelung*, Leipzig, 4th edition, 1901; *Encyklop. d. mathem. Wissenschaften*, Vol. IV. Heft 1, Leipzig, 1901.

emphasise this point, because the conditional equations, expressing the connections, can also be of a different kind, namely *vectorial*. Let, for instance, one of the particles be constrained to remain on a given straight line: this condition can be expressed by two scalar equations, for instance by the equations of two planes intersecting along that straight line; these two equations will deprive the particle of two degrees of freedom. But we can do this in a simpler way, viz. by writing down a single vector equation

$$\mathbf{r} = x\mathbf{a} + \mathbf{b},$$

where $\mathbf{a}$ is a constant vector taken in the direction of the straight line in question, $\mathbf{b}$ any other constant vector (which runs from O to any arbitrary point chosen once and for ever on the given straight line), and x a freely variable scalar. Then we shall get

$$\delta\mathbf{r} = \mathbf{a}\,\delta x,$$

which is the most immediate expression of the condition requiring the particle to remain on the given straight line. This one vector equation deprives the particle of as many degrees of freedom (*i.e.* two) as did the two scalar equations mentioned before. But these are, obviously, pure questions of form. Further on, when considering the equations of Lagrange (in their 'first' form), it will be convenient to express all the connections of the system in the scalar form, *i.e.* by scalar functions of vectors. But meanwhile the choice of the form is quite indifferent.

As to the definition of virtual displacements, the following further circumstance must be emphasised here:

> The conditional equations, *i.e.* the equations expressing the connections, may contain the time t explicitly. If this be the case, we call *virtual only those* displacements which satisfy these equations, *after we have put in them* $t = const.$

From this additional explanation of the definition it follows, for instance, that the real displacements $d\mathbf{r}$ occurring during the motion of the system, in the time interval dt, are virtual displacements when and only when the connections do not involve the time explicitly.

To give an example, if one of the particles is constrained to remain always on the surface of a sphere (of radius R), whose centre C is moved about in space in a given way, the corresponding conditional equation will contain t explicitly. In fact, denoting by $\mathbf{l}$ the vector drawn from the fixed point O to the centre C, at the time being (Fig. 26), we have as the expression of the supposed constraint:

$$(\mathbf{r} - \mathbf{l})^2 = R^2.$$

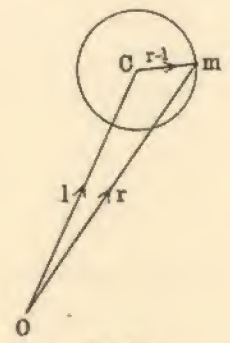

FIG. 26.

Now, $\mathbf{l}$ being an explicit function of the time, this condition contains *explicitly* the variable t. In this case, then, a *virtual* displacement of the particle (m) will be a displacement $\delta\mathbf{r}$, which leaves it on the sphere supposed to be *stopped*, *i.e.* in which $\mathbf{l}$ is treated as a constant; thus, the condition for $\delta\mathbf{r}$ will be

$$(\mathbf{r} - \mathbf{l})\,\delta\mathbf{r} = 0,$$

or $\delta\mathbf{r} \perp$ to the vector $\mathbf{r} - \mathbf{l}$, in the instantaneous position of the sphere's centre. In this case the real displacement

$$d\mathbf{r} = \mathbf{v}\,dt$$

will generally *not* be contained among the virtual displacements, for it may have a component (not only normal but also) parallel to the vector $\mathbf{r} - \mathbf{l}$.

Having thus explained the precise meaning of the virtual displacements $\delta\mathbf{r}_1$, $\delta\mathbf{r}_2$, etc., generally $\delta\mathbf{r}$, we can now write down the *principle of d'Alembert* in the vector form.

> This principle, expressed briefly in the old-fashioned manner, is that *the virtual work of all the lost forces is equal to zero.**

The virtual work of the 'lost force' $\mathbf{S} = m\ddot{\mathbf{r}} - \mathbf{F}$ is the product of the absolute value (or tensor) of the virtual displacement and of the component of $\mathbf{S}$ along this displacement, or, by the definition of the scalar product of two vectors, is equal to $\mathbf{S}\,\delta\mathbf{r}$. The 'virtual work' of all the 'lost forces' $\mathbf{S}_1$, $\mathbf{S}_2$, etc., is the algebraic sum of similar

* For its history see *Encyklop. d. math. Wiss.*, loc. cit. p. 78.

products for all the particles of the system, *i.e.* $\mathbf{S}_1\delta\mathbf{r}_1 + \mathbf{S}_2\delta\mathbf{r}_2 + \dots$ or, more shortly, $\Sigma\mathbf{S}\,\delta\mathbf{r}$.

Thus, the vector expression of **d'Alembert's Principle** is

$$\Sigma(m\ddot{\mathbf{r}} - \mathbf{F})\,\delta\mathbf{r} = 0, \qquad (1)$$

where the summation extends to all the particles.

Henceforth, deducing from (1) particular corollaries, and also transformations not less general than (1), we shall try to travel by the purely vectorial road, *i.e.* without recurring to any artificial splitting of forces, or of accelerations or of the virtual displacements themselves, into components, rectangular or other.

Lagrange's Equations.*

These equations are wholly equivalent to d'Alembert's Principle, *i.e.* they express the same thing in a different form.

When the system is free, the displacements $\delta\mathbf{r}$ are perfectly arbitrary and independent of one another, so that it follows immediately from (1) that for every particle

$$m\ddot{\mathbf{r}} - \mathbf{F} = 0,$$

as in (*a*) above. The equations (*a*) are already Lagrange's equations for a *free* system.

But let the system be constrained. Let its connections be expressed in a finite form (*holonomic* system) by κ mutually independent equations

$$\phi = 0, \quad \psi = 0, \quad \chi = 0, \text{ etc.}, \qquad (2)$$

where ϕ, ψ, χ, etc., are *scalar* functions of the positions of all, or of some, particles of the system, which functions may also contain explicitly the time t.

We have no need to trouble ourselves whether the positions of the particles 1, 2, etc., on which ϕ, ψ, etc., depend in a given manner, are expressed by rectangular or polar, or any other co-ordinates of the particles, or by the above $\mathbf{r}$'s. It is sufficient to know that ϕ, for instance, depends in some given way on the position of the particle 1, and say also on the position of 2, 3, 4, etc., *i.e.* that ϕ changes its value when one or more of these particles change their position in space. The reader may imagine, if he wishes, that ϕ is from the beginning given as some scalar function of the vectors $\mathbf{r}_1, \mathbf{r}_2, \dots$; or, if it is not given in this form, he may imagine it to have been reduced to this form. But for the general

* Of the 'first' form.

consideration of our subject all such questions are completely indifferent.

It is sufficient to know that ϕ, and similarly ψ, etc., vary in a manner which depends on the positions of the particles 1, 2, and so on.

Now, if ϕ depended only on the position of *one* particle, we should denote the gradient or *slope* of this function by $\nabla\phi$, and if $\mathbf{r}$ be the vector corresponding to this particle, then, from the condition $\phi = \text{const.} = 0$, would follow

$$\nabla\phi \,.\, \delta\mathbf{r} = 0$$

as the condition for the virtual displacement $\delta\mathbf{r}$.

But, as ϕ may depend on the positions of several particles, 1, 2, etc. (generally i), of which the corresponding vectors are $\mathbf{r}_1$, $\mathbf{r}_2$, etc. (generally $\mathbf{r}_i$), we cannot of course speak simply about '*the* gradient' of this function ϕ, but we must specify also the particle, relatively to whose displacement in space the gradient has to be taken. For this purpose we shall add to the symbol ∇ the index $_{(1)}$, or $_{(2)}$, or generally $_{(i)}$, so that

$$\nabla_{(1)}\phi, \quad \nabla_{(2)}\phi, \text{ generally } \nabla_{(i)}\phi,$$

will be the gradients of ϕ corresponding to change of position of the particle 1 only, or 2 only, or i only, while all the remaining particles are considered to be fixed. These gradients could be called **partial gradients**, in the same way as we speak of partial differential quotients in ordinary, scalar analysis. Each of the partial gradients $\nabla_{(i)}\phi$ is, of course, a vector, namely a vector normal to that surface which is expressed by $\phi = \text{const.}$, on the supposition that all the particles, implied in ϕ, are fixed with the exception of the ith particle.

Using this notation, we get from $\phi = 0$ the following condition for the virtual displacements:

$$\nabla_{(1)}\phi \,.\, \delta\mathbf{r}_1 + \nabla_{(2)}\phi \,.\, \delta\mathbf{r}_2 + \dots = 0$$

or, written shortly, $\quad \Sigma\nabla_{(i)}\phi \,.\, \delta\mathbf{r}_i = 0,$

where the summation extends to all the particles of the system. Similarly for the remaining conditions $\psi = 0$, $\chi = 0$, and so on.

Hence, by considering all the κ connections (2) imposed on the system, we get as many equations for the virtual displacements:

$$(\kappa \text{ equations}) \quad \left\{\begin{array}{l} \Sigma\nabla_{(i)}\phi \,.\, \delta\mathbf{r}_i = 0 \\ \Sigma\nabla_{(i)}\psi \,.\, \delta\mathbf{r}_i = 0 \\ \dots\dots\dots\dots \\ \dots\dots\dots\dots \end{array}\right\}. \qquad (3)$$

To satisfy d'Alembert's principle (1) and also the conditional equations (3), the well-known method of Lagrangian indeterminate multipliers can be used. Multiplying, then, the equations (3) respectively by the scalar coefficients λ, μ, etc., then adding them to the equation (1) and equating to zero the coefficient of each $\delta\mathbf{r}_i$ separately, we get the required **Lagrangian equations of motion**

$$m\ddot{\mathbf{r}}_i = \mathbf{F}_i + \lambda\nabla_{(i)}\phi + \mu\nabla_{(i)}\psi + \dots . \tag{4}$$

The number of these equations is equal to the number of particles, say n. We have then n *vector* differential equations (4) and κ *scalar* equations (2) for the n vectors $\mathbf{r}_1, \mathbf{r}_2, \dots \mathbf{r}_n$ and the κ scalar multipliers $\lambda, \mu, \dots$. (In Cartesian language we should say: we have $3n$ scalar differential equations and κ equations expressing the connections, that is $3n+\kappa$ equations for the $3n$ coordinates and the κ multipliers $\lambda, \mu, \dots$.)

The equations (4) are of the 2nd order with respect to the time t; if then for the initial instant $t = t_0$ the positions and the velocities, *i.e.* the vectors $(\mathbf{r}_1)_0, (\mathbf{r}_2)_0, \dots ; \ (\dot{\mathbf{r}}_1)_0, (\dot{\mathbf{r}}_2)_0, \dots$,

be given, the whole motion of the system, or its time-history, will be determined, on the supposition, of course, that the forces $\mathbf{F}_i$ are given functions of the positions and velocities of the particles constituting the system. These forces, as well as the connections, may contain the time t explicitly. The reader will observe that in deducing (3) from (2) the time has not been varied, according to the observation made above in regard to the complete definition of virtual displacements.

As an *example* of the application of (4), take the case of a system consisting of a *single* particle constrained to remain on the surface

$$\phi = 0. \tag{2a}$$

In this case (4) gives

$$m\ddot{\mathbf{r}} = \mathbf{F} + \lambda\nabla\phi. \tag{4a}$$

As the vector $\nabla\phi$ is normal* to the surface in question, the additional force

$$\mathbf{N} = N\mathbf{n} = \lambda\nabla\phi = \lambda\frac{\partial\phi}{\partial n}\mathbf{n}, \tag{5}$$

expressing the 'reaction' of the surface, is *normal* to the surface. To find its intensity

$$N = \lambda\frac{\partial\phi}{\partial n}, \tag{5'}$$

we have only to determine the Lagrangian coefficient λ from the given condition $\phi = 0$.

* By (x.), Chapter I.

Let, for instance, the surface $\phi = 0$ be a *sphere* at rest, of radius R and centre O; then

$$\phi = \tfrac{1}{2}(r^2 - R^2) = 0, \tag{2b}$$

so that

$$\nabla\phi = \mathbf{n}\frac{\partial\phi}{\partial\mathbf{r}} = \mathbf{n}r = \mathbf{n}R = \mathbf{r}. \tag{6}$$

Hence, the Lagrangian equation of motion will in this case become

$$m\ddot{\mathbf{r}} = \mathbf{F} + \lambda\mathbf{r}. \tag{4b}$$

To find λ, write simply, according to (v.),

$$r^2 = \mathbf{r}^2 = \text{const.} = R^2,$$

and differentiate with regard to t; then

$$\mathbf{r}\dot{\mathbf{r}} = 0,$$

and differentiating again:

$$\mathbf{r}\ddot{\mathbf{r}} + \dot{\mathbf{r}}^2 = 0.$$

Now substitute $\ddot{\mathbf{r}}$ from (4b); then

$$0 = \mathbf{F}\mathbf{r} + \lambda\mathbf{r}^2 + m\dot{\mathbf{r}}^2 = R\mathbf{F}\mathbf{n} + \lambda R^2 + mv^2;$$

hence

$$\lambda R = -\mathbf{F}\mathbf{n} - \frac{mv^2}{R}, \tag{7}$$

where $\mathbf{v} = \dot{\mathbf{r}}$ is the instantaneous velocity of the particle.

From (7) we get, by (5) and (6), the final expression of the force of reaction

$$\mathbf{N} = \lambda\mathbf{r} = -\mathbf{n}\left(\mathbf{F}\mathbf{n} + \frac{mv^2}{R}\right). \tag{8}$$

Remembering that $\mathbf{F}\mathbf{n}$ is the normal (or radial) component of the impressed force $\mathbf{F}$, the reader himself will translate the formula (8) into physical language and recognise it as a well-known proposition of elementary dynamics.

To get from (4b) the differential equation of motion of a simple *pendulum*, of length R, we have only to write

$$\mathbf{F} = mg\mathbf{a}, \tag{9}$$

where g is the 'terrestrial acceleration' and $\mathbf{a}$ denotes a vertical unit vector, pointing downwards.

It will be observed that multiplying scalarly the Lagrangian equations (4) by the corresponding $\delta\mathbf{r}_i$'s and adding them, we get, by (3), the Principle of d'Alembert, which was our point of departure. Thus, the equations of Lagrange, with given equations of the connections of the system, are wholly equivalent to d'Alembert's Principle.

Hamilton's Principle.

This also is equivalent to d'Alembert's Principle. The vectorial road, again, leads very easily from the second to the first, and *vice versa*.

Denoting the virtual work of all impressed forces $\mathbf{F}$ by $\delta' W$,* *i.e.* writing, for the sake of brevity,

$$\delta' W = \Sigma \mathbf{F} \delta \mathbf{r}, \tag{10}$$

the Principle of d'Alembert becomes

$$\Sigma m \ddot{\mathbf{r}} \delta \mathbf{r} = \delta' W. \tag{1'}$$

Now, for each particle separately,

$$\frac{d}{dt}(\dot{\mathbf{r}}\,\delta\mathbf{r}) = \ddot{\mathbf{r}}\,\delta\mathbf{r} + \dot{\mathbf{r}}\frac{d}{dt}\delta\mathbf{r} = \ddot{\mathbf{r}}\,\delta\mathbf{r} + \dot{\mathbf{r}}\,\delta\frac{d\mathbf{r}}{dt}$$

$$= \ddot{\mathbf{r}}\,\delta\mathbf{r} + \mathbf{v}\,\delta\mathbf{v};$$

$$\therefore\ \ddot{\mathbf{r}}\,\delta\mathbf{r} = \frac{d}{dt}(\mathbf{v}\,\delta\mathbf{r}) - \tfrac{1}{2}\delta(v^2),$$

where $\mathbf{v} = \dot{\mathbf{r}}$ is the velocity of the particle in question. Multiply both sides by its mass m and sum up for all the particles of the system; then, by (1'),

$$\delta \Sigma \tfrac{1}{2} m v^2 + \delta' W = \frac{d}{dt} \Sigma m \mathbf{v}\,\delta\mathbf{r},$$

or, denoting the *vis-viva* or *kinetic energy* of the whole system by T, *i.e.* putting $\frac{1}{2}\Sigma m v^2 = T$:

$$\delta T + \delta' W = \frac{d}{dt} \Sigma m \mathbf{v}\,\delta\mathbf{r};$$

hence, by integration from any instant $t = a$ to any other arbitrary instant $t = b$:

$$\int_a^b (\delta T + \delta' W)\, dt = [\Sigma m \mathbf{v}\,\delta\mathbf{r}]_a^b, \tag{11}$$

where $[\ \ ]_a^b$ denotes, as usual, $[\ \ ]_{t=b} - [\ \ ]_{t=a}$.

Now, if the displacements $\delta\mathbf{r}$, virtual but otherwise arbitrary during the interval of time $a \to b$, *vanish for the terminal instants* a, b *themselves*, the right side of equation (11) is $= 0$, so that

$$\int_a^b (\delta T + \delta' W)\, dt = 0, \tag{12}$$

and this equation (together with the condition of vanishing $\delta\mathbf{r}$'s for the instants a, b) is an expression of what is called **Hamilton's Principle.**

*We write $\delta' W$ (and *not* δW) to emphasise that this infinitesimal work is, generally, *not* a complete variation of a function of position and time.

Observing that the instants a and b can be chosen in an arbitrary manner, the reader will easily pass back from Hamilton's to d'Alembert's Principle.

The deduction, from (12), of the so-called 'second' form of Lagrange's equations of motion is a purely *scalar* question, and can therefore be omitted here. Moreover, it is accomplished in a few lines by introducing the configurational coordinates $q_1, q_2, \dots q_s$ (s = number of degrees of freedom) and the corresponding velocities $\dot{q}_1, \dot{q}_2, \dots \dot{q}_s$, and integrating by parts the terms $\frac{\partial T}{\partial \dot{q}_i}\delta\dot{q}_i$, so that $-\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)\delta q_i$ come in instead; in this way, writing $\delta' W = \Sigma Q_i \delta q_i$, so that Q_i is the (generalised) force corresponding to q_i, the reader will get immediately from (12):

$$Q_i = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i}\ (i = 1, 2, \dots s), \tag{A}$$

which is the well-known second form of Lagrange's equations of motion.

Still one remark. In Hamilton's Principle (12) not a single heavy (clarendon) letter occurs, which circumstance is the most evident sign of its scalarity; all space-directional properties have disappeared, and this is what constitutes the advantage of Hamilton's Principle, inasmuch as it permits us to introduce directly *any* configurational variables or independent *parameters* equal in number to the degrees of freedom of the system, without regard to the particular type of its connections.

CHAPTER III.

SPECIAL PRINCIPLES.

LET us now consider three principles or propositions, less general than d'Alembert's Principle, which follow from this principle under particular conditions.

1. The Principle of Vis-viva.

If the equations (2), or the 'connections' of the system, **do not contain the time t explicitly**, then among the virtual displacements $\delta\mathbf{r}_i$ are also contained the actual displacements $d\mathbf{r}_i$ of the particles, occurring during their motion in time, viz. in the time-element dt. For a system, then, which satisfies this condition we can put in d'Alembert's Principle (1):

$$\delta\mathbf{r}_i = \dot{\mathbf{r}}_i\, dt = \mathbf{v}_i\, dt,$$

where dt has *the same* value for all particles $i = 1, 2$, etc. Hence, writing $\ddot{\mathbf{r}} = \dot{\mathbf{v}}$, and omitting the common factor dt, we get

$$\Sigma\mathbf{F}\mathbf{v} = \Sigma m\mathbf{v}\dot{\mathbf{v}} = \frac{d}{dt}\Sigma\tfrac{1}{2}mv^2,$$

or, denoting again the kinetic energy of the system by T,

$$\frac{dT}{dt} = \Sigma\mathbf{F}\mathbf{v} = \Sigma\mathbf{F}\dot{\mathbf{r}}. \tag{13}$$

This equation, true for any instant t, can be read as follows:

The increase of kinetic energy of the system, per unit time, is equal to the activity of the impressed forces or to the work done by them on the system, per unit time.

Integrating both sides, we have

$$T_b - T_a = W_{ab}, \tag{14}$$

where W_{ab} is the work done on the system in the interval from $t = a$ to $t = b$.

This is the **principle of Vis-viva.**

In particular, if the forces $\mathbf{F}_i$ have a (scalar) *potential* independent of t, *i.e.* if

$$\mathbf{F}_i = \nabla_{(i)} U \tag{15}$$

and $\partial U/\partial t = 0$, then

$$\Sigma\mathbf{F}_i\dot{\mathbf{r}}_i = \Sigma\frac{d\mathbf{r}_i}{dt}\nabla_{(i)}U = \frac{dU}{dt};$$

$\therefore$ by (13):

$$\frac{d}{dt}(T - U) = 0 \tag{13a}$$

or $T - U = const.$ In other words, the function $T - U$ is an **invariant** of the system.

$-U$ is called the *potential energy* and $T - U$ the *total energy* or, shortly, *the* energy of the system. Thus, under the conditions stated, the energy is an invariant of the system, or rather *one* of its invariants, inasmuch as a system of more than one degree of freedom has several essential, *i.e.* mutually independent, invariants. A system of s degrees of freedom (in the common, mechanical sense of the word) has $2s - 1$ essential invariants. They cannot all be found with the same ease,—but that does not matter here.

If the potential U is a one-valued function of the position of the particles, then T recovers its value every time that all the particles pass through their original positions. This property is called the **principle of conservation of Vis-viva.**

Note. If the connections of the system *do contain the time explicitly*, then the equation (13) is not satisfied; to obtain the more general equation which then *is* true, multiply Lagrange's equations (4) scalarly by the corresponding $\mathbf{v}_i$'s and add them together; then

$$\frac{dT}{dt} = \Sigma\mathbf{F}_i\mathbf{v}_i + \lambda\Sigma\mathbf{v}_i\nabla_{(i)}\phi + \mu\Sigma\mathbf{v}_i\nabla_{(i)}\psi + \dots; \tag{a}$$

but, by total differentiation of the conditions $\phi = 0$, $\psi = 0$, etc., with respect to t,

$$\frac{\partial\phi}{\partial t} + \Sigma\mathbf{v}_i\nabla_{(i)}\phi = 0, \quad \frac{\partial\psi}{\partial t} + \Sigma\mathbf{v}_i\nabla_{(i)}\psi = 0, \text{ etc.}$$

$$\therefore \frac{dT}{dt} = \Sigma\mathbf{F}\mathbf{v} - \lambda\frac{\partial\phi}{\partial t} - \mu\frac{\partial\psi}{\partial t} - \dots, \tag{16}$$

which is the required equation, *satisfied for any system.*

If $\phi, \psi, \dots$ do not contain the time explicitly, we have

$$\frac{\partial\phi}{\partial t} = 0, \quad \frac{\partial\psi}{\partial t} = 0, \text{ etc.},$$

and (16) reduces immediately to (13).

The reader ~~must~~ need not be surprised to see the increase of kinetic energy, in (16), not to be equal to the work of the *impressed* forces. Go a step backwards, to (α), and the reason will appear immediately. For this purpose it will be sufficient to consider a system consisting of a single particle subjected to one constraint only, say, a particle constrained to remain on the surface $\phi = 0$ which is *moved* about in space in any *given* way, so that ϕ contains t explicitly. Then (α) reduces to

$$\frac{dT}{dt} = \mathbf{F}\mathbf{v} + \lambda \mathbf{v}\nabla\phi ;$$

but, returning to the equation of motion $m\ddot{\mathbf{r}} = \mathbf{F} + \lambda\nabla\phi$, we see that $\mathbf{N} = \lambda\nabla\phi$, as in (5), is the force of 'reaction' of the surface. Hence, the above equation may be written

$$\frac{dT}{dt} = \mathbf{F}\mathbf{v} + \mathbf{N}\mathbf{v} = (\mathbf{F} + \mathbf{N})\mathbf{v},$$

or in words: the increase of kinetic energy *is equal to the work done by the total force*, impressed *plus* reactive.

Observe that $\mathbf{N}$ is *always normal* to the surface $\phi = 0$, be it fixed or moved. Now, if the surface is fixed, then $\mathbf{v}$ is tangent to it, and $\mathbf{N}\mathbf{v}$ vanishes, so that the 'reaction' does no work; but if the surface be moved, then $\mathbf{v}$ is generally not tangential, and $\mathbf{N}$ *does* work, the activity of $\mathbf{N}$ being

$$\mathbf{N}\mathbf{v} = \lambda \mathbf{v}\nabla\phi = -\lambda \frac{\partial\phi}{\partial t}.$$

If, for instance, the moved surface be a sphere, then (as in a previous example, to which corresponds the Fig. 26)

$$\phi = \tfrac{1}{2}(\mathbf{r} - \mathbf{l})^2 - \tfrac{1}{2}K^2 = 0,$$

and

$$\frac{\partial\phi}{\partial t} = -(\mathbf{r} - \mathbf{l})\frac{\partial \mathbf{l}}{\partial t} = -K\mathbf{n}\mathbf{v}_c,$$

where $\mathbf{v}_c$ is the velocity of the centre C of the sphere. Now, $\mathbf{n}\mathbf{v}_c$ or the normal component of the velocity of C is generally different from zero, and so also will be the supplementary activity,

$$\mathbf{N}\mathbf{v} = \lambda K \mathbf{N}\mathbf{v}_c.$$

2. The Principle of Centre of Gravity.*

If the connections be (none or) such that it is possible to displace all the particles of the system simultaneously *by one and the same*

*=centre of mass.

length $\delta\epsilon$ *in one and the same direction* $\mathbf{a}$, or, in mathematical language, if it is allowed to put

$$\delta\mathbf{r}_1 = \delta\mathbf{r}_2 = \ldots = \delta\mathbf{r}_n = \mathbf{a}\,\delta\epsilon,$$

where $\mathbf{a}$ is some unit-vector and $\delta\epsilon$ an infinitesimal scalar, it follows from d'Alembert's Principle (1), by omitting the common factor $\delta\epsilon$, that

$$\mathbf{a}\Sigma(m\ddot{\mathbf{r}} - \mathbf{F}) = 0. \qquad (a)$$

Here $\mathbf{F}\mathbf{a} = \mathbf{a}\mathbf{F}$ is the component of the force $\mathbf{F}$ along $\mathbf{a}$; denote it by F_a and write

$$\Sigma m\mathbf{r} = M\mathbf{S}, \quad M = \Sigma m, \qquad (17)$$

also S_a for the component of the vector $\mathbf{S}$ along $\mathbf{a}$; then, by (a),

$$M\ddot{S}_a = M\frac{d^2S_a}{dt^2} = \Sigma F_a. \qquad (18a)$$

The point defined by (17), *i.e.* the end of the vector $\mathbf{S}$, is called the **centre of gravity** or, more correctly, the *centre of mass* of the whole system; S_a is its coordinate measured along $\mathbf{a}$, the initial point of $\mathbf{S}$, as of all the $\mathbf{r}$'s being O. Bearing this in mind, the reader will easily recognise (18a) as the expression of *the principle of motion of the centre of mass, for the direction* $\mathbf{a}$.

It is scarcely necessary to say that the position of the centre of mass is independent of the choice of the point of reference O, or of any auxiliary framework (or system of coordinates), and that it depends exclusively on the distribution of mass in the system.* If this distribution be continuous in space, the summation Σ becomes an integration:

$$M = \int dm = \int \rho\, d\tau, \quad M\mathbf{S} = \int \mathbf{r}\rho\, d\tau,$$

ρ being the density of mass and $d\tau$ an element of volume.

If some other direction $\mathbf{b}$ has the same property as $\mathbf{a}$, we have not only (a), but also

$$\mathbf{b}\Sigma(m\ddot{\mathbf{r}} - \mathbf{F}) = 0 \qquad (b)$$

or

$$M\ddot{S}_b = \Sigma F_b.$$

But then this property belongs also to any direction $\mathbf{k}$ *parallel to the plane* $\mathbf{a}$, $\mathbf{b}$; in fact, any such $\mathbf{k}$ can be put into the form

$$\mathbf{k} = x\mathbf{a} + y\mathbf{b},$$

where x, y are scalars; hence we have only to multiply the equation (a) by x and (b) by y, and add, in order to obtain

$$\mathbf{k}\Sigma(m\ddot{\mathbf{r}} - \mathbf{F}) = 0.$$

*The vectorial proof of this statement is left to the reader as an exercise.

If, finally, a third direction **c**, *not coplanar* with **a**, **b**, also possesses the property of **a**, **b**, we have

$$\mathbf{c}\Sigma(m\ddot{\mathbf{r}} - \mathbf{F}) = 0 \qquad (c)$$

or

$$M\ddot{S}_c = \Sigma F_c.$$

Then, and only then, it follows, from the three equations (*a*), (*b*) and (*c*), that *the whole vector* $\Sigma(m\ddot{\mathbf{r}} - \mathbf{F})$ vanishes, so that

$$M\ddot{\mathbf{S}} = M\frac{d^2\mathbf{S}}{dt^2} = \Sigma\mathbf{F}, \qquad (18)$$

which is called **the principle of motion of the centre of mass**, without any reservation, *i.e.* for *all* directions in space.

In particular, if all the impressed forces are in mutual equilibrium, *i.e.* if $\Sigma\mathbf{F} = 0$, or—in other terms—if the vectors **F**, arranged in a chain, form a *closed* polygon,

$$\ddot{\mathbf{S}} = 0; \quad \therefore \ \mathbf{S} = \mathbf{A}t + \mathbf{B}, \qquad (19)$$

A and **B** being constant vectors. Under these conditions the centre of mass has a uniform motion along a straight line. This is the principle of **conservation** of the motion of the centre of mass.

3. The Principle of Areas.

If the connections of the system permit *every particle of it to be simultaneously turned through the same angle* $\delta\theta$ *about the same axis* **a**, *i.e.* if (taking the point of reference *O* on *this* axis)

$$\delta\mathbf{r}_i = \delta\theta . \mathrm{V}\mathbf{a}\mathbf{r}_i, \quad i = 1, 2, \ldots n$$

is a virtual displacement, then introducing it in d'Alembert's Principle (1), and omitting the common factor $\delta\theta$, which is a simple scalar, we get

$$\Sigma(m\ddot{\mathbf{r}}\mathrm{V}\mathbf{a}\mathbf{r} - \mathbf{F}\mathrm{V}\mathbf{a}\mathbf{r}) = 0.$$

Now, by (VIII.), $\mathbf{F}\mathrm{V}\mathbf{a}\mathbf{r} = \mathbf{a}\mathrm{V}\mathbf{r}\mathbf{F}$, and similarly $\ddot{\mathbf{r}}\mathrm{V}\mathbf{a}\mathbf{r} = \mathbf{a}\mathrm{V}\mathbf{r}\ddot{\mathbf{r}}$, and as the axis of rotation **a** is, by supposition, the same for all the particles of the system, it can be put before Σ, so that

$$\mathbf{a}\Sigma(m\mathrm{V}\mathbf{r}\ddot{\mathbf{r}} - \mathrm{V}\mathbf{r}\mathbf{F}) = 0.$$

The vector sum

$$\Sigma\mathrm{V}\mathbf{r}\mathbf{F}$$

is the resultant **moment**, about *O*, of the impressed forces, as regards both direction and intensity; denote it by **L**. Then, remembering that $\mathrm{V}\dot{\mathbf{r}}\dot{\mathbf{r}} = 0$, identically, we have

$$\frac{d}{dt}\mathrm{V}\mathbf{r}\mathbf{v} = \frac{d}{dt}\mathrm{V}\mathbf{r}\dot{\mathbf{r}} = \mathrm{V}\mathbf{r}\ddot{\mathbf{r}},$$

so that the last equation takes the form

$$\mathbf{a}\left\{\frac{d}{dt}\Sigma m\mathrm{V}\mathbf{r}\mathbf{v} - \mathbf{L}\right\} = 0. \qquad (a)$$

Similarly, if the whole system can be turned also about another axis **b** passing through *O*, and also about a third axis **c** passing through *O* and not coplanar with **a**, **b**, we have

$$\mathbf{b}\{\text{as above}\} = 0, \qquad (b)$$
$$\mathbf{c}\{\text{as above}\} = 0, \qquad (c)$$

and from the three equations (*a*), (*b*) and (*c*) it follows that the bracketed vector expression must vanish, or that

$$\frac{d}{dt}\Sigma m\mathrm{V}\mathbf{r}\mathbf{v} = \mathbf{L}. \qquad (20)$$

The vector product $\frac{1}{2}\mathrm{V}\mathbf{r}\mathbf{v}$ expresses,* by its tensor, the *area* swept out by the vector **r** in unit time in the plane **r**, **v**, its direction being normal to this plane. Whence the name **principle of areas** given to what is expressed by (20).

It is *the* principle of areas, without reservation, *i.e.* for any conceivable axis of rotation.

The equation (*a*) alone, for example, is the principle of areas *belonging to the axis* **a**, viz. for areas swept out in the plane normal to **a**.

The vector sum

$$\Sigma m\mathrm{V}\mathbf{r}\mathbf{v}$$

is what is called **the moment of momentum** of the whole system, about *O*.

Thus, (20) may be enunciated:

The rate of time variation of the moment of momentum is equal, in absolute value and direction, to the resultant moment of the impressed forces, both moments being taken about the same point *O*, which, of course, can be chosen in a perfectly arbitrary manner. In particular, if $\mathbf{L} = 0$, we have **the conservation of areas**, *i.e.*

$$\frac{d}{dt}\Sigma\ldots = 0$$

or

$$\Sigma m\mathrm{V}\mathbf{r}\mathbf{v} = \mathbf{C}, \qquad (21)$$

the vector **C** being constant both in its tensor and direction. The plane normal to this vector **C** is called the **invariable plane** of the

* As shown in an example, given in Chapter I.; see page 27.

system. If the system consists of a single particle, then its orbit lies in this plane; in this case (21) reduces to

$$\mathrm{V}\mathbf{rv} = \mathbf{const.}$$

(Kepler's II. law, for instance).

If not the whole moment $\mathbf{L}$, but only some component of it, say $\mathbf{La}$ or L_a, vanishes, the principle of conservation of areas applies only to the axis $\mathbf{a}$, *i.e.* to the areas swept out in the plane normal to $\mathbf{a}$. In such a case we have only one *scalar invariant*, viz. $\mathbf{a}\Sigma m\mathrm{V}\mathbf{rv} = const.$

But if $\mathbf{L}=0$, as above, we have (21), *i.e.* one *vector invariant*, which is equivalent to *three scalar invariants*.

The principle of conservation of areas, in its full extent, holds, for instance, for a system of free particles acted on by 'central forces,' *i.e.* by forces $\mathbf{F}_i$ coinciding in direction with the corresponding $\mathbf{r}_i$'s, because then for each particle separately $\mathrm{V}\mathbf{rF}=0$, and hence $\mathbf{L}=0$.

Kepler's second law, already alluded to, is the same thing as conservation of areas for a single particle, viz. a planet in its motion round the sun. Its equivalence to 'centrality' of motion has been already shown in Chapter I.

To illustrate the simultaneous validity of the principle of conservation of areas and of vis-viva, let us consider a single particle, not subjected to any constraint and acted on by a force $\mathbf{F}$, not only *central, i.e.* directed towards or from the fixed centre O, but also such that its intensity is a function of *the distance* r *alone* (r being the tensor of $\mathbf{r}$), say $F = F(r)$.

The unit-vector drawn from O to the particle, at the time being, may be written $\mathbf{r}/r$, hence

$$\mathbf{F} = \mathbf{r}\frac{F(r)}{r}.$$

Now, put $$\int F(r)\,dr = U = U(r),^*$$

so that $$F(r) = \frac{dU}{dr};$$

then U will be the *potential* of the force $\mathbf{F}$, *i.e.*

$$\mathbf{F} = \nabla U.$$

In fact, since U depends only on r and since ∇U means simply the slope of U, we have, by Chapter I.,

$$\nabla U = \frac{\mathbf{r}}{r}\frac{dU}{dr} = \mathbf{r}\frac{F(r)}{r} = \mathbf{F}. \quad \text{Q.E.D.}$$

* The additive constant of integration being unessential.

Thus, our force $\mathbf{F}$ *has a potential*, and since it is besides a *central force*, both the principles mentioned above hold, *i.e.*

$$\tfrac{1}{2}mv^2 - U(r) = c, \qquad (\alpha)$$

$$\mathrm{V}\mathbf{rv} = \mathbf{a}. \qquad (\beta)$$

where c is a *scalar constant*, namely the total energy, and $\mathbf{a}$ a *vector constant*; its tensor, a, is twice the area swept out by the radius vector, per unit time; and since also the direction of $\mathbf{a}$ is constant, the vectors $\mathbf{r}$, $\mathbf{v}$ are always in the same plane containing O, *i.e.* the particle describes a *plane orbit* round O. If the revolution round O, in the plane of Fig. 27, be clockwise, then $\mathbf{a}$ is normal to the paper, pointing away *from* the reader. In other words, $\mathbf{a}$, $\mathbf{r}$, $\mathbf{v}$ is a *right-handed* system.

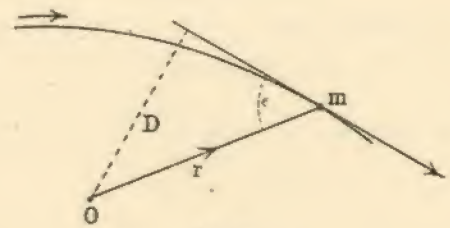

FIG. 27.

To obtain the equation of the orbit, we have to eliminate the velocity v from (α), (β). Multiply both sides of (β) vectorially by $\mathbf{v}$; then, by the fundamental formula (IX.),

$$\mathrm{V}\mathbf{va} = \mathrm{V}\mathbf{v}\mathrm{V}\mathbf{rv} = v^2\mathbf{r} - (\mathbf{vr})\mathbf{v};$$

multiply this scalarly by $\mathbf{r}$, getting thus

$$\mathbf{r}\mathrm{V}\mathbf{va} = v^2r^2 - (\mathbf{vr})^2,$$

or, since by (VIII.) $$\mathbf{r}\mathrm{V}\mathbf{va} = \mathbf{a}\mathrm{V}\mathbf{rv} = a^2,$$

$$v^2r^2 - (\mathbf{vr})^2 = a^2.$$

If $\mathbf{l}$ be a unit-vector tangent to the orbit, drawn in the direction of motion, then $\mathbf{v} = v\mathbf{l}$, and consequently

$$v^2\{r^2 - (\mathbf{rl})^2\} = a^2.$$

Now, if ϵ be the angle included by $\mathbf{r}$ and $\mathbf{l}$,

$$\mathbf{rl} = r\cos\epsilon, \quad r^2 - (\mathbf{rl})^2 = r^2\sin^2\epsilon;$$

hence $vr\sin\epsilon = a$ or

$$vD = a = \text{const.}, \qquad (\beta')$$

where D is the length of the perpendicular from O upon the tangent. Thus, the velocity is inversely proportional to this perpendicular,—

which is the known *theorem of Newton*, holding *for any central motion*. Observe that (β') has been deduced uniquely from (β), without recurring to (α), *i.e.* to the existence of a potential.

Combining (β') with (α), the required elimination of v is effected immediately, leading to

$$\frac{ma^2}{2D^2} - U(r) = c, \qquad (\gamma)$$

which already *is* the equation of *the orbit*, for *any law* of dependence of the central force on the distance r, *i.e.* for any form of the function $F(r)$.

In particular, if this be the *Newtonian law of gravitation*, say

$$F = -\frac{Mm}{r^2}$$

(where M is a constant), then

$$U = \int F\,dr = \frac{Mm}{r},$$

and the equation of the orbit, (γ), becomes

$$\frac{a^2}{2D^2} - \frac{M}{r} = \frac{c}{m},$$

where everything but D, r is constant. The proof that this is a *conic*, with one of its foci in O, may be left to the reader.

CHAPTER IV.

RIGID DYNAMICS.

General Remarks.

For studying the motion of a rigid system, let the vector $\mathbf{r} = O \to m$ determine, as in the preceding chapters, the instantaneous position of a material particle m of the system in question, relatively to the point O, which can be regarded as a point belonging to some 'fixed' system of reference* (see Fig. 28). The instantaneous velocity of

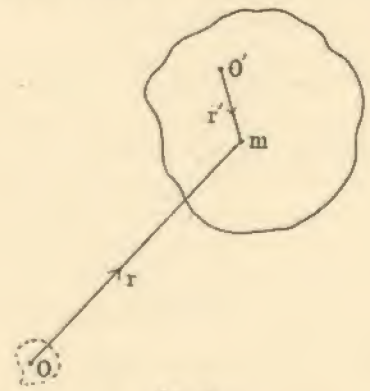

Fig. 28.

the particle m, also relative to O, will be, as before,

$$\mathbf{v} = d\mathbf{r}/dt = \dot{\mathbf{r}}.$$

* The reason being that in order to determine a vector $\mathbf{r}$ in absolute value and *direction*, a single *point* of reference O is not sufficient; to do this, some extended, non-symmetrical system is necessary, to which O may belong as one of its points. This is the reason why in Fig. 28, and also in Fig. 25, round the point O has been drawn a (dotted) contour, representing such a reference *system*. In the course of our reasoning we shall often refer to the '*point O*,' but always in the sense here explained.

We shall also choose once and for ever some point O' fixed *in the rigid system itself*, as the initial point of a system of reference invariably attached to the rigid system, and we shall denote the vector $O' \to m$ by $\mathbf{r}'$. The vector $\mathbf{r}'$ will thus determine the position of the particle m as seen by an observer attached to the rigid system, while $\mathbf{r}$ will do the same office for an observer attached to O. In other words, the vector $\mathbf{r}'$ will distinguish the individual material particle m of the rigid system from all other particles of this system.

Thus, the *rigidity* of the system to be considered in this chapter will be completely expressed by saying that $\mathbf{r}'$ does not vary with the time, or by writing

$$\dot{\mathbf{r}}' = \frac{d\mathbf{r}'}{dt} = 0.$$

For a rigid system which is perfectly free to move about in space (having 6 degrees of freedom) all the three principles treated in Chap. III. hold: 1° the principle of vis-viva, because the connections do not contain the time t explicitly; 2° the principle of centre of mass, because the whole system can be moved (displaced) in any direction; 3° the principle of areas, because the rigid system can be turned as a whole about any axis.

Differential Equations of purely Rotational Motion.

As the treatment of translatory motion of the whole system does not offer any difficulties, we shall consider here only the case of *purely rotational* motion of the rigid system *about a 'fixed' point*, *i.e.* about a point which does not move relatively to the point O. The rigid system will then have only *three* degrees of freedom. The principles of vis-viva and of areas hold, but the principle of centre of mass will lose its applicability, since one of the points of the rigid system is fixed.

Let O' be this fixed point of the system. We can in this case, if we wish, let O coincide with O', but then we must not forget that the dotted contour surrounding the point O (Fig. 28, as explained above) will not keep company with the rigid system in its rotational motion. In other words, notwithstanding that the points O, O' are made to coincide with one another, the vectors $\mathbf{r}$, $\mathbf{r}'$, though overlapping in space, will behave *differently in time.*

We shall still have

$$\dot{\mathbf{r}}' = 0,$$

while

$$\dot{\mathbf{r}} = \mathbf{v} \neq 0, \text{ in general;}$$

but $\mathbf{r}^2$ or r^2, and hence also r, will be constant in time for every

point of the rigid system, r being simply its *scalar distance* from O, or, what is the same, from O'.

Just as with $\mathbf{r}$, $\mathbf{r}'$, we shall also denote *any vector*, say $\mathbf{w}$, in a twofold way, viz.

by $\mathbf{w}$, if it is treated in regard to the fixed system of reference (O), and by $\mathbf{w}'$, if it is taken in regard to the moving system (O'), *i.e.* in regard to the rotating rigid system.

It will be convenient to refer to these two ways of treating a vector by saying shortly: 'relatively to O,' in the first, and 'relatively to O',' in the second case.

Let now $\mathbf{p}$ be the *instantaneous angular velocity* of the rigid system, in absolute value and direction (*i.e.* so that p is the absolute value of the angular velocity, and the direction of the vector $\mathbf{p}$ is in the *positive* sense of the axis of rotation*). Then, considering any vector $\mathbf{w}$, and its *alter ego* $\mathbf{w}'$, and observing that $\dot{\mathbf{w}} - \dot{\mathbf{w}}'$ is simply the resultant velocity with which the end-point of the vector $\mathbf{w}$ moves on account of the rotation of the rigid system alone, we have the important relation

$$\dot{\mathbf{w}} - \dot{\mathbf{w}}' = \mathrm{V}\mathbf{pw} \qquad (22)$$

or

$$d\mathbf{w}/dt = d\mathbf{w}'/dt + \mathrm{V}\mathbf{pw}.\dagger$$

The formula for the velocity $\mathbf{v}$ of a point of the rigid system,

$$\mathbf{v} = \dot{\mathbf{r}} = \mathrm{V}\mathbf{pr}, \qquad (23)$$

is only a particular case of the more general formula (22); for we have only to write in (22) $\mathbf{r}$ instead of $\mathbf{w}$ and remember that $\dot{\mathbf{r}}' = 0$, to get immediately (23).

After these, slightly lengthy but necessary, preliminaries we pass to the dynamics of a rigid system.

We can apply to it at once the *principle of areas*, as expressed by the formula (20), Chap. III.

Denoting the moment of momentum of the system about O by $\mathbf{q}$, *i.e.* writing

$$\mathbf{q} = \Sigma m \mathrm{V}\mathbf{rv}, \qquad (24)$$

and calling, as before, the moment of the impressed forces $\mathbf{L}$, we have instead of (20)

$$\frac{d\mathbf{q}}{dt} = \mathbf{L}. \qquad (25)$$

* Viz. in that direction in which a person, looking along the axis, would find the rotation about it to be right-handed or clockwise.

† This is the vector form of what in Routh's *Advanced Rigid Dynamics* is called a 'Fundamental Theorem,' the proof of which covers there one page and a half. This, by the way only, to show the convenience of vector language, as compared with scalar or Cartesian.

This is the differential equation for $\mathfrak{q}$ relative to O, *i.e.* relative to the 'fixed' system of reference. Applying the general formula (22), we get from it immediately the differential equation for $\mathfrak{q}'$, relative to the rotating rigid system itself:

$$\frac{d\mathfrak{q}'}{dt} = \mathrm{V}\mathfrak{q}\mathfrak{p} + \mathfrak{L}. \tag{26}$$

The moment of the impressed forces $\mathfrak{L}$ is given, and between the moment of momentum $\mathfrak{q}$ and the angular velocity $\mathfrak{p}$ there exists at every instant a certain relation based only on the properties of the rigid system, which also can be considered as given; this relation will be developed in the next section.

Taking into account this relation between $\mathfrak{q}$ and $\mathfrak{p}$, peculiar to the given system, we have ultimately in (26) *one* vector differential equation of the first order for *one* vector, namely for $\mathfrak{q}$ if $\mathfrak{p}$ be eliminated, and *vice versa*. If then $\mathfrak{p}$, say $=\mathfrak{p}_0$, be given for any particular instant, $t=t_0$, then the whole motion of the system will also be determined.

Kinetic Energy. Principal Axes. Euler's Equations of Motion.

By (23),

$$v^2 = \mathfrak{v}\mathrm{V}\mathfrak{p}\mathfrak{r} = \mathfrak{p}\mathrm{V}\mathfrak{r}\mathfrak{v};$$

hence the *kinetic energy* $T=\frac{1}{2}\Sigma m v^2$ of the rotating rigid system will be

$$T = \tfrac{1}{2}\mathfrak{p}\Sigma m\mathrm{V}\mathfrak{r}\mathfrak{v},$$

or, by (24),

$$T = \tfrac{1}{2}\mathfrak{p}\mathfrak{q}, \tag{27}$$

i.e. equal to *half the scalar product of the angular velocity and the moment of momentum.*

Now for the mutual dependence of $\mathfrak{p}$ and $\mathfrak{q}$, promised at the end of the preceding section. This is also a very simple matter.

By (23),

$$\begin{aligned}\mathrm{V}\mathfrak{r}\mathfrak{v} &= \mathrm{V}\mathfrak{r}\mathrm{V}\mathfrak{p}\mathfrak{r}\\ &= r^2\mathfrak{p} - (\mathfrak{p}\mathfrak{r})\mathfrak{r}, \text{ by (IX.)};\end{aligned}$$

hence, by (24),

$$\mathfrak{q} = \mathfrak{p}\Sigma m r^2 - \Sigma m(\mathfrak{p}\mathfrak{r})\mathfrak{r}. \tag{28}$$

Here both the members on the right side contain $\mathfrak{p}$ to the *first* power.

Thus, the moment of momentum $\mathfrak{q}$ is a **linear vector function** of the angular velocity $\mathfrak{p}$, and this, *i.e.* (28), in fact, *is* already the required relation. Denoting by K the operator implied in (28), which turns $\mathfrak{p}$ to $\mathfrak{q}$, we can write shortly

$$\mathfrak{q} = K\mathfrak{p}. \tag{28a}$$

K is called a **linear vector operator.** It changes not only the tensor, but in general also the direction of the vector which is operated on.

The linear vector operator has found its chief application in the researches of electromagnetic phenomena in crystals; $\mathfrak{q}$ is related to $\mathfrak{p}$ just as the electric displacement is related to the electric force, or as the magnetic induction to the magnetic force.*

In the case before us the reader has to look on the operator K as being, for the time, only a short symbol for certain operations which are fully determined by the preceding formula (28). Further on we shall see that, in fact, all the properties of the operator K can be deduced from this formula.

Introducing (28*a*) in (27), the kinetic energy of the rigid system can be expressed by

$$T = \tfrac{1}{2}\mathfrak{p}K\mathfrak{p}, \tag{27a}$$

i.e. as a (quadratic) function of the angular velocity alone.

In the same way, $\mathfrak{q}=K\mathfrak{p}$ can be substituted in the equation of motion (26). From (28) we see at a glance that all properties of the operator K depend only on the mass of the rigid system and on its distribution about O; whence it follows that K does not vary with time, so that it can be written before the differentiator d/dt. We have, then, instead of (26):

$$K\frac{d\mathfrak{p}'}{dt} = \mathfrak{L} - \mathrm{V}\mathfrak{p}K\mathfrak{p}, \tag{26a}$$

and this form of the equation of motion is the vectorial incarnation of the three celebrated (scalar) **equations of Euler,** as will be seen explicitly further on, by an appropriate decomposition of (26*a*).

Meanwhile let us look for the properties of the operator K.

In the first place, then, what are the principal axes of the operator K, or in mechanical terminology the **principal axes of the rigid system,** *i.e.* what are the directions $\pm\mathfrak{x}$, for which the moment of momentum $\mathfrak{q}$ coincides with the axis of instantaneous rotation,

* For the theory and application of the linear vector operator see, for instance, Heaviside's *Electromagnetic Theory*, Vol. I., Chap. III. Linear vector functions were first introduced by Hamilton in his calculus of quaternions.

It may be remarked here that the above operator K, which converts $\mathfrak{p}$ into $\mathfrak{q}$, is a *symmetrical* (or self-conjugate) operator. The more general, *non-symmetrical* linear vector operators, which have nothing to do with *rigid* dynamics, will appear in the next chapter as very useful for the treatment of problems connected with non-rigid bodies.

or with the direction of $\mathbf{p}$? This fundamental question can be answered easily.

We may consider $\mathbf{x}$ as a *unit* vector, so that $\mathbf{p}=p\mathbf{x}$, $\mathbf{q}=q\mathbf{x}$, and the required principal axes will have the property

$$\mathbf{q}=n\mathbf{p}=np\mathbf{x}, \tag{A}$$

where n is a *scalar*.

Introducing (A) in (28) and dividing both sides by p, we get

$$\mathbf{x}\Sigma mr^2-\Sigma m(\mathbf{r}\mathbf{x})\mathbf{r}=n\mathbf{x}$$

or

$$\Sigma m(\mathbf{r}\mathbf{x})\mathbf{r}=\mathbf{x}(\Sigma mr^2-n). \tag{B}$$

If $+\mathbf{x}$ satisfies (B), $-\mathbf{x}$ will also, with the same value of n. Hence a principal axis can be denoted shortly by writing simply $\mathbf{x}$ instead of $\pm\mathbf{x}$.

Suppose now that $\mathbf{x}_1$ and $\mathbf{x}_2$ are two different principal axes, to which correspond the particular values n_1, n_2 of the scalar n. Then, by (B),

$$\Sigma m(\mathbf{r}\mathbf{x}_1)\mathbf{r}=\mathbf{x}_1(\Sigma mr^2-n_1), \tag{B$_1$}$$

$$\Sigma m(\mathbf{r}\mathbf{x}_2)\mathbf{r}=\mathbf{x}_2(\Sigma mr^2-n_2), \tag{B$_2$}$$

whence, multiplying, scalarly, (B$_1$) by $\mathbf{x}_2$ and (B$_2$) by $\mathbf{x}_1$ and subtracting,

$$(n_2-n_1)\mathbf{x}_1\mathbf{x}_2=0.$$

Hence, if $n_1\neq n_2$, then $\mathbf{x}_1\mathbf{x}_2=0$, *i.e.*

> Two principal axes with *different* values of n are *perpendicular* to one another.

Hence, if $\mathbf{x}_3$ be a third principal axis to which belongs a value $n=n_3$ *different* from both n_1 and n_2, this axis $\mathbf{x}_3$ will again be *perpendicular* to the first two axes, so that $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ will constitute a *normal* system of unit vectors. We shall choose the order of the indices 1, 2, 3 so that $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ is a *right-handed* system, *i.e.*

$$\mathbf{x}_1=V\mathbf{x}_2\mathbf{x}_3, \quad \mathbf{x}_2=V\mathbf{x}_3\mathbf{x}_1, \quad \mathbf{x}_3=V\mathbf{x}_1\mathbf{x}_2,$$

as for $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$.

Generally, n_1, n_2, n_3 will be different from one another. But if it happens that, say, $n_2=n_1$, then multiplying (B$_1$) by any scalar a and (B$_2$) by any other scalar b and adding,

$$\Sigma m\mathbf{r}[\mathbf{r}(a\mathbf{x}_1+b\mathbf{x}_2)]=(a\mathbf{x}_1+b\mathbf{x}_2)(\Sigma mr^2-n_1),$$

so that the vector $a\mathbf{x}_1+b\mathbf{x}_2$ also satisfies (B). Hence:

1° If $\mathbf{x}_1$, $\mathbf{x}_2$ are two principal axes having *equal* values of n, then *any vector* $(a\mathbf{x}_1+b\mathbf{x}_2)$ taken at random *in the plane* $\mathbf{x}_1$, $\mathbf{x}_2$ *is also a principal axis, with the same value of* n.

And from this proposition follows immediately:

2° If *three*, non-coplanar, principal axes have *equal* values of n then *any* direction in space, drawn from O, is also a principal axis with same value of n, *i.e.* the directions of $\mathbf{q}$ and $\mathbf{p}$ coincide for any direction of $\mathbf{p}$.

But if to three principal axes belong three *different* values of n, it is obvious that *no other* principal axes can exist, as a fourth principal axis would, by the above properties, be identical with either $\mathbf{x}_1$, $\mathbf{x}_2$ or $\mathbf{x}_3$.

Hence, in the most general case, a rigid system has *three principal axes* $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ *mutually perpendicular*, to which correspond different values n_1, n_2, n_3 of the scalar n.*

By this reasoning we see that the vector equation (B) admits in the most general case *only three* different roots n corresponding to principal axes. Indeed, it is not difficult to deduce from (B) a *cubic* scalar equation for n. But we can find all roots from the equation (B) itself.

Let us denote the components of $\mathbf{r}$ along $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ by r_1, r_2, r_3, or write

$$\mathbf{r}\mathbf{x}_1=r_1, \quad \mathbf{r}\mathbf{x}_2=r_2, \quad \mathbf{r}\mathbf{x}_3=r_3;$$

more generally, let w_1, w_2, w_3 be the components of any vector $\mathbf{w}$ along $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$. (It may be observed that *this* is not an *artificial* decomposition of vectors into scalars; for $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ represent essentially peculiar directions, and hence properties belonging intrinsically to the given rigid system, and are no more artificial than are, say, the optical axes of a crystal.)

Now, substituting in (B), successively, $\mathbf{x}=\mathbf{x}_1$, and so on, we get

$$\left.\begin{matrix}\Sigma mr_i\mathbf{r}=\mathbf{x}_i(\Sigma mr^2-n_i)\\ i=1,\ 2,\ 3\end{matrix}\right\}; \tag{C}$$

* If $n_1=n_2\neq n_3$, we have the above case 1°, or **axial symmetry** as regards rotational inertia; in this case, the third axis only, *i.e.* $\mathbf{x}_3$, corresponding to n_3, is usually called *the principal axis* of the rigid system.

If $n_1=n_2=n_3$, we have the above case 2°, viz. complete **isotropy** as regards rotational inertia; in this case, *all* directions having the same property, *none* of them is called (in the accepted terminology) a principal axis. In this case the *operator* K degenerates into an *ordinary scalar factor*.

then, multiplying respectively by $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ and remembering that

$$\mathbf{x}_1^2=\mathbf{x}_2^2=\mathbf{x}_3^2=1, \quad \mathbf{r}^2=r_1^2+r_2^2+r_3^2,$$
$$n_1=\Sigma m(r^2-r_1^2)=\Sigma m(r_2^2+r_3^2), \text{ etc.},$$

or, finally,

$$n_1=\Sigma m\rho_1^2, \quad n_2=\Sigma m\rho_2^2, \quad n_3=\Sigma m\rho_3^2, \tag{D}$$

ρ_1, ρ_2, ρ_3 being the shortest distances of a particle m of the system from the three principal axes.

At the same time it follows from (C), by remembering that $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ are mutually perpendicular,

$$\Sigma mr_2r_3=\Sigma mr_3r_1=\Sigma mr_1r_2=0.$$

This is a well-known property of the principal axes, and determines their directions if the distribution of mass be given.

The scalar quantities n_1, n_2, n_3 (D) are called **the principal moments of inertia** of the rigid system about the point O. As they are, at the same time, **the principal values of the linear vector operator** K, we may conveniently denote them by K_1, K_2, K_3. Remember that these are ordinary scalars. The operator K could be called the **inertial operator** of the given rigid system (the point O being tacitly assumed). If the structure of K is known, all that regards the dynamics of rotation of the particular system is given.

Returning to (A), we can write

$$q_1=K_1p_1, \quad q_2=K_2p_2, \quad q_3=K_3p_3$$

or, vectorially,

$$\mathbf{q}=K\mathbf{p}=\mathbf{x}_1K_1p_1+\mathbf{x}_2K_2p_2+\mathbf{x}_3K_3p_3, \tag{29}$$

where

$$K_1=\Sigma m\rho_1^2, \quad K_2=\Sigma m\rho_2^2, \quad K_3=\Sigma m\rho_3^2. \tag{30}$$

Also $\rho_1^2=(\mathrm{V}\mathbf{r}\mathbf{x}_1)^2$, and so on.

The above equation may also be inverted, thus

$$\mathbf{p}=K^{-1}\mathbf{q},$$

the operator K^{-1} being the so called **inverse** of K, *i.e.* also a linear vector operator having the same principal axes as K and the principal values

$$K_1^{-1}, \quad K_2^{-1}, \quad K_3^{-1},$$

as is seen immediately from the developed right side of (29).

Remembering that $\mathbf{x}_1^2=1$, etc., $\mathbf{x}_2\mathbf{x}_3=0$, etc., we can write, by (29), the kinetic energy expressed by (27*a*) in the form

$$T=\tfrac{1}{2}(K_1p_1^2+K_2p_2^2+K_3p_3^2). \tag{27b}$$

Substituting (29) in (26*a*), splitting this vectorial equation into three scalar ones, along the principal axes, and remembering that $\mathrm{V}\mathbf{x}_2\mathbf{x}_3=\mathbf{x}_1$, etc., we get

$$\left.\begin{aligned} K_1\frac{dp_1}{dt}&=(K_2-K_3)p_2p_3\\ K_2\frac{dp_2}{dt}&=(K_3-K_1)p_3p_1\\ K_3\frac{dp_3}{dt}&=(K_1-K_2)p_1p_2 \end{aligned}\right\}, \tag{26b}$$

which is the usual form of *Euler's equations* of motion. The components p_1, p_2, p_3 of the angular velocity being already taken along the (moving) principal axes of the rigid system itself, no dashes are necessary.

The equation (26*a*), *i.e.*

$$K\frac{d\mathbf{p}'}{dt}=\mathbf{L}+\mathrm{V}\mathbf{q}\mathbf{p},$$

with $\mathbf{q}=K\mathbf{p}$, is the complete vectorial equivalent of the three scalar equations of Euler.

Multiplying this vector equation, scalarly, by $\mathbf{p}'=\mathbf{p}$ and remembering that $\mathbf{p}\mathrm{V}\mathbf{q}\mathbf{p}=0$, identically, we have

$$\mathbf{p}'K\frac{d\mathbf{p}'}{dt}=\mathbf{L}\mathbf{p};$$

but by (27*a*),

$$\frac{dT}{dt}=\frac{dT'}{dt}=\frac{1}{2}\left(\mathbf{p}'K\frac{d\mathbf{p}'}{dt}+\frac{d\mathbf{p}'}{dt}K\mathbf{p}'\right)=\mathbf{p}'K\frac{d\mathbf{p}'}{dt},*$$

hence

$$\frac{dT}{dt}=\mathbf{L}\mathbf{p}; \tag{31}$$

and $\mathbf{L}\mathbf{p}$ being the activity of the impressed forces, this is the expression of *the principle of vis-viva*. In fact, as we saw at the beginning, this principle is true for any rigid system.

* Since $\frac{d\mathbf{p}'}{dt}K\mathbf{p}'=\mathbf{p}'K\frac{d\mathbf{p}'}{dt}$, and, in general, for any pair of vectors $\mathbf{A}$, $\mathbf{B}$,

$$\mathbf{A}K\mathbf{B}=\mathbf{B}K\mathbf{A},$$

provided that the linear vector operator K be a *symmetrical* operator, as in our case it *is*. In fact, $\mathbf{A}K\mathbf{B}=A_1K_1B_1+A_2K_2B_2+A_3K_3B_3=B_1K_1A_1+\ldots=\mathbf{B}K\mathbf{A}$. In the next chapter it will be seen that this property does *not* belong to *non-symmetrical* operators.

Motion under no Forces (Motion à la Poinsot).

If there are no impressed forces or, at least, if their resultant moment $\mathbf{L}$ vanishes (for the given fixed point O'), it follows immediately from (25) that

$$\frac{d\mathbf{q}}{dt}=0, \tag{32}$$

i.e. the moment of momentum $\mathbf{q}$ *has a constant value and constant direction 'in space,' i.e.* relatively to the system of reference O (the system of 'fixed' stars or any other moving relatively to it with uniform translational velocity). The plane normal to this invariable direction is called the **invariable plane** of the rigid system.

Euler's equations of motion, or rather their vectorial equivalent (26*a*), reduce in this case to

$$K\frac{d\mathbf{p}'}{dt}=\frac{d\mathbf{q}'}{dt}=V\mathbf{qp}, \tag{33}$$

where $\mathbf{q}=K\mathbf{p}$, as before.

If the rigid system rotates at a given instant about one of its principal axes, *i.e.* if $\mathbf{q}=n\mathbf{p}$, then $V\mathbf{qp}=0$ and, by (33),

$$\frac{d\mathbf{p}'}{dt}=0,$$

i.e. the system will continue to rotate for ever about this same principal axis with constant angular velocity.

Besides the principal axes no other axis of rotation has this important property. In fact, $V\mathbf{qp}$ vanishes only if $\mathbf{q}\parallel\mathbf{p}$ (the trivial case of $p=q=0$ * not being worthy of notice). On account of this important property, the principal axes are also called the **free axes** or **the permanent axes of rotation** of the rigid system.

The eq. (33), multiplied scalarly by $\mathbf{p}=\mathbf{p}'$, or also (31) for $\mathbf{L}=0$, gives

$$\frac{dT}{dt}=0,$$

i.e.
$$T=\tfrac{1}{2}\mathbf{p}K\mathbf{p}=\tfrac{1}{2}(K_1p_1^2+K_2p_2^2+K_3p_3^2)=\text{const.} \tag{34}$$

Thus, the kinetic energy is now an *invariant* of the rigid system, which could have been foreseen.

Again, (33) multiplied scalarly by $\mathbf{q}=\mathbf{q}'$ gives $\mathbf{q}'d\mathbf{q}'/dt=0$

or
$$\mathbf{q}^2=(K\mathbf{p})^2=K_1^2p_1^2+K_2^2p_2^2+K_3^2p_3^2=\text{const.}, \tag{35}$$

but this is not an independent invariant, since we had already, by (32), $\mathbf{q}=\mathbf{const.}$, whence also $q^2=const.$ or $q=\text{const.}$

* *i.e.* of a *non-rotating* rigid system.

We must remember that $\mathbf{q}$ is constant, but not $\mathbf{q}'$, which generally varies with the time, *i.e.* the moment of momentum is fixed 'in space,' but variable relatively to the rigid system itself.

Thus, we have ultimately, in the case of $\mathbf{L}=0$:

1° as the expression of conservation of areas, *one vector invariant* $\mathbf{q}$ equivalent to *three scalar invariants*;

2° as the expression of conservation of vis-viva, or kinetic energy, *one scalar invariant* $T=\tfrac{1}{2}\mathbf{pq}=\tfrac{1}{2}\mathbf{p}K\mathbf{p}$;

therefore, in all, *four* scalar invariants.

Now, the system, having three degrees of freedom, admits in all

$$2\cdot3-1=5 \text{ scalar invariants}$$

(essential, *i.e.* mutually independent). Out of the five we have, then, in the case of no impressed moment, four invariants. But, in this case, the knowledge of the above four invariants is already sufficient to reduce the problem of integration to simple quadratures. (See the end of this section.)

The property 1°, expressing the conservation of areas or, better, the conservation of moment of momentum, determines *the invariable plane*. Now, by 2° or by $\mathbf{pq}=const.$, the projection of the vector $\mathbf{p}$ on $\mathbf{q}$ is constant; hence:

The end-point of the vector $\mathbf{p}$ *moves in the invariable plane* (Fig. 29).

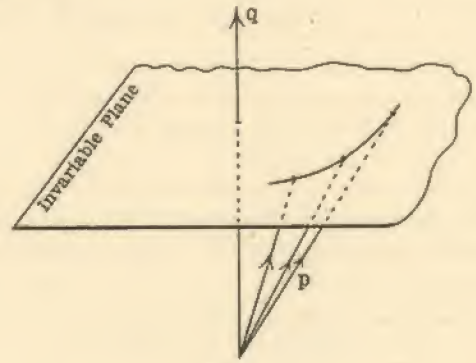

Fig. 29.

Again, writing 2° in the form (34), we see that the 'locus geometricus' of this end-point of $\mathbf{p}$ is

the ellipsoid $T=\tfrac{1}{2}\mathbf{p}K\mathbf{p}=const.$;

this is called **the ellipsoid of inertia** * or 'momental ellipsoid.' It is rigidly attached to the system; its axes fall in the principal axes of the system, their lengths being inversely proportional to the square roots of the corresponding principal moments of inertia; by (34), the semiaxes of this surface are, respectively,

$$\sqrt{\frac{2T}{K_1}},\quad \sqrt{\frac{2T}{K_2}},\quad \sqrt{\frac{2T}{K_3}}.$$

Thus, the shape of this ellipsoid depends on the properties of the rigid system itself, and its size on the given constant value of kinetic energy. To imply both, this surface may be called *the ellipsoid $T=const.$* or, simply, *the ellipsoid T*. This ellipsoid, then, is, for $L=0$, the locus of the end-point of the vector $\mathbf{p}$.

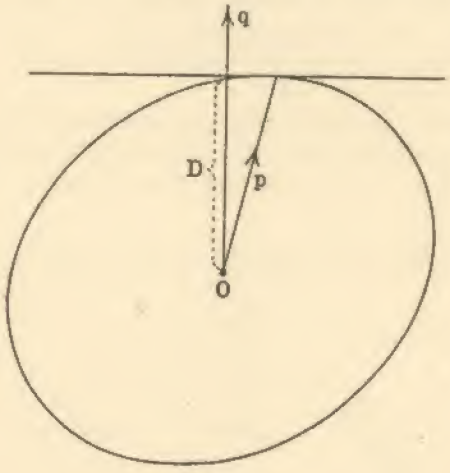

FIG. 30.

On the other hand, denoting by T_p the kinetic energy regarded as a homogeneous quadratic function of the component angular velocities p_1, p_2, p_3, we have

$$2T_p = p_1\frac{\partial T_p}{\partial p_1} + p_2\frac{\partial T_p}{\partial p_2} + p_3\frac{\partial T_p}{\partial p_3}$$

or

$$2T_p = \mathbf{p}\nabla T_p,$$

* Properly speaking, this name belongs to the ellipsoid

$$K_1p_1^2 + K_2p_2^2 + K_3p_3^2 = 1,$$

that is, for $2T=1$. But, the choice of units being arbitrary, this is an indifferent matter.

where the operator ∇ has here in the three-dimensional manifold p_1, p_2, p_3 the same meaning as the common ∇ in ordinary space.

But $2T_p = 2T = \mathbf{pq}$, for *all* values of $\mathbf{p}$ and of the corresponding $\mathbf{q}$;

$$\therefore\ \mathbf{q} = \nabla T_p, \tag{36}$$

whence it is seen at once that *the vector $\mathbf{q}$ is normal to the plane which touches the ellipsoid T in the end-point of the vector $\mathbf{p}$* (see Fig. 30).

From the same formula (36) it follows also that

$$q = \frac{2T}{D}, \tag{37}$$

where D is the length of the perpendicular from the centre of the ellipsoid to the above tangent plane.* This is the usual geometrical representation of the relation of any two vectors, one of which is a linear (symmetrical) function of the other, like $\mathbf{p}$ and $\mathbf{q}=K\mathbf{p}$.

Thus, the ellipsoid T, which is invariably attached to the rigid system, always touches the invariable plane; but the point of tangency is the extremity of the axis of instantaneous rotation, and has therefore no velocity relatively to this plane, *i.e.* does not glide on it.

Hence, in the case of no impressed moment ($L=0$), the motion of the rigid system is such that *the ellipsoid $T=const.$, with its centre fixed, rolls on the invariable plane.*

The discovery of this beautiful property is due to Poinsot. Whence also the motion of a rigid system about a fixed point under no impressed forces is usually called *Poinsot's motion* or motion *à la Poinsot.*

The most general motion of this kind can be represented analytically, as depending on the time and on the initial conditions, by elliptic functions.

To obtain dp/dt as a function of p itself and of the given values of the invariants T and q,—which is the starting point for the analytical solution of the problem alluded to, namely for the expression of t

* In fact, by purely geometrical reasons,

$$D = \mathbf{p}\nabla T_p : \frac{\partial T_p}{\partial n},\ \text{where}\ \nabla = \mathbf{n}\frac{\partial}{\partial n},\ \text{as in (x.)};$$

but, by (36),

$$\mathbf{q} = \nabla T_p\ \text{or}\ q = \frac{\partial T_p}{\partial n};\quad \therefore\ qD = \mathbf{p}\nabla T_p = 2T\ \text{or}\ q = 2T/D.$$

as a function of p and then, by inversion, of p as an (elliptic) function of the time t,—go back to the equation of motion (33),

$$K\frac{d\mathbf{p}'}{dt} = \mathrm{V}\mathbf{q}\mathbf{p}, \text{ say} = \mathbf{w}$$

or

$$\frac{d\mathbf{p}'}{dt} = K^{-1}\mathbf{w},$$

where K^{-1} or $1/K$ is the inverse of K, as explained above. Multiply, scalarly, both sides of the last equation by $\mathbf{p}' = \mathbf{p}$, and remember that

$$\mathbf{p}'\,d\mathbf{p}'/dt = \tfrac{1}{2}\,d(p'^2)/dt = \tfrac{1}{2}\,d(p^2)/dt = p\frac{dp}{dt}.$$

Then

$$p\frac{dp}{dt} = \mathbf{p}K^{-1}\mathbf{w} = (K^{-1}\mathbf{p})\mathbf{w} = (K^{-1}\mathbf{p})\mathrm{V}\mathbf{q}\mathbf{p},$$

since K^{-1}, exactly as K itself, is a symmetrical operator.* But

$$(K^{-1}\mathbf{p})\mathrm{V}\mathbf{q}\mathbf{p} = \mathbf{p}\mathrm{V}(K^{-1}\mathbf{p})\mathbf{q},$$ by (VIII.), and $\mathbf{q} = K\mathbf{p}$; hence:

$$dt = \frac{p\,dp}{\mathbf{p}\mathrm{V}(K^{-1}\mathbf{p})(K\mathbf{p})},$$

or, if the time t be counted from the instant for which $p = p_0$,

$$t = \int_{p_0}^{p} \frac{p\,dp}{\mathbf{p}\mathrm{V}(K^{-1}\mathbf{p})(K\mathbf{p})}, \qquad (\alpha)$$

which is the required expression.

The denominator, if appropriately developed (see '*Appendix*'), proves to be the *square root of a cubic function of* p^2 alone, so that t is expressed by an **elliptic integral**; whence, by inversion, the scalar angular velocity p reduces to **elliptic functions** of the time t.

It may be observed here that the integration according to (α) (or rather to its developed form, given in the 'Appendix') has been effected in finite terms in two cases, namely when $K_2 = K_3$, $K_1 > K_2$, *i.e.* for an *uniaxial* rigid system, and when $2T/q^2 = K_2$ for the case of *three* principal axes, K_2 being neither the greatest nor the least of the three principal moments of inertia.

Rotation of a Rigid System under the Action of Gravity.

Let, again, any point O' of the rigid system be fixed (but not its centre of gravity, since in this case we should have $L = 0$, and hence again Poinsot's motion, already considered in the last section).

*Observe that the transposition of K is allowed only for the *scalar product, i.e.* $\mathbf{A}K\mathbf{B}$ or $\mathbf{A}(K\mathbf{B}) = (K\mathbf{A})\mathbf{B}$, but *not* for the vector product; in fact, $\mathrm{V}\mathbf{A}K\mathbf{B}$ or $\mathrm{V}\mathbf{A}(K\mathbf{B})$ is a quite different thing from $\mathrm{V}(K\mathbf{A})\mathbf{B}$. Consequently $\mathrm{V}(K^{-1}\mathbf{p})(K\mathbf{p})$ *is not at all equal to* $\mathrm{V}\mathbf{p}\mathbf{p}$, which would be nil.

The reference point O may again coincide with O'. We shall adhere also to all the previous notation.

The moment $\mathbf{L}$ of the impressed forces $\mathbf{F}$ has been in general

$$\mathbf{L} = \Sigma\mathrm{V}\mathbf{r}\mathbf{F}.$$

If then the only force acting on every particle of the rigid system (or element of a continuous rigid body) is its own *weight*

$$\mathbf{F} = mg\mathbf{a},$$

we have

$$\mathbf{L} = \Sigma mg\mathrm{V}\mathbf{r}\mathbf{a}.$$

Here $\mathbf{a}$ is a unit vector, directed vertically downwards, and hence fixed relatively to the system of reference O, and g the acceleration of terrestrial gravity. For simplicity we shall not take into account the earth's rotation, so that the system of reference attached to the earth will be equivalent to the 'fixed' system of reference. Moreover, we shall, of course, suppose that the dimensions of our rigid system are negligible in comparison with those of the earth, so that the scalar g will have the same numerical value and the unit vector $\mathbf{a}$ the same direction for all parts of the rigid system.

Under these conditions,

$$\mathbf{L} = g\mathrm{V}(\Sigma m\mathbf{r})\mathbf{a},$$

or, if $M = \Sigma m$ be the whole mass of the system and $\mathbf{S}$ the vector which determines the instantaneous position of the centre of gravity C relatively to O [as in formula (17)]:

$$\mathbf{L} = gM\mathrm{V}\mathbf{S}\mathbf{a},$$

or, finally writing $\mathbf{S} = S\mathbf{s}$, *i.e.* denoting by S the (scalar) distance of

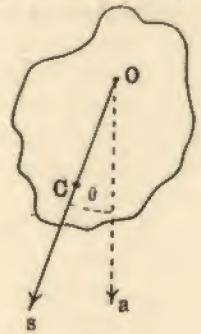

FIG. 31.

the centre of gravity C from the fixed point O and by $\mathbf{s}$ a *unit* vector pointing from O towards C (Fig. 31):

$$\mathbf{L} = c\mathrm{V}\mathbf{s}\mathbf{a}, \qquad (38)$$

where

$$c = MgS \qquad (39)$$

is a scalar constant. We must remember that $\mathbf{a}$ is fixed in space (*i.e.* relatively to the earth), while $\mathbf{s}$ is rigidly attached to the rigid system and rotates together with it.

Substituting (38) in the general equations of motion, (25) and (26), we have for our heavy rigid system

$$\frac{d\mathbf{q}}{dt} = c\mathrm{V}\mathbf{sa} \tag{40}$$

and

$$\frac{d\mathbf{q}'}{dt} = \mathrm{V}\mathbf{qp} + c\mathrm{V}\mathbf{sa}, \tag{40'}$$

where, as before, $\mathbf{q} = K\mathbf{p}$, K being the inertial operator of the system, about O, as explained above.

Multiply (40), scalarly, by $\mathbf{a}$; then, remembering that $\mathbf{a}\mathrm{V}\mathbf{sa} = 0$ and that $d\mathbf{a}/dt = 0$, we get

$$\frac{d}{dt}(\mathbf{qa}) = 0. \tag{41}$$

Precisely the same thing would follow from (40'), by taking into account the general relation (22). Eq. (41) expresses the conservation of areas in the horizontal plane, or the conservation of the vertical component of the moment of momentum. For planes other than horizontal, the principle of conservation of areas obviously does not hold in the present case.

As the force of gravity has a (scalar) potential, the principle of conservation of vis-viva must hold. And, indeed, multiplying (40') by $\mathbf{p} = \mathbf{p}'$ or using the equation (31) obtained previously, we have

$$\frac{dT}{dt} = c\mathbf{p}\mathrm{V}\mathbf{sa} = c\mathbf{a}\mathrm{V}\mathbf{ps};$$

but $\mathbf{s}$ is fixed *in the rigid system*, or

$$\frac{d\mathbf{s}'}{dt} = 0;$$

hence, by (22): $$\mathrm{V}\mathbf{ps} = \frac{d\mathbf{s}}{dt}; \quad \frac{dT}{dt} = c\mathbf{a}\frac{d\mathbf{s}}{dt},$$

and, as $\mathbf{a}$ is fixed relatively to the earth, *i.e.* $d\mathbf{a}/dt = 0$, and as c is a scalar constant, we get ultimately

$$\frac{d}{dt}(T - c\,\mathbf{as}) = 0. \tag{42}$$

Writing $c = MgS$, by (39), and remembering that $\mathbf{as} = \cos(\mathbf{a}, \mathbf{s})$, it is not difficult to recognise in the product $-c\mathbf{as}$ the usual

expression for potential energy, *i.e.* the weight of the system × height of its centre of gravity. Thus, (42) expresses the constancy of the sum of kinetic and potential energy, or the constancy of the total energy of the *whole* system (consisting of the rotating rigid system and the attracting earth).

By (41) and (42) we have, then, *two scalar invariants*:

$$\mathbf{qa} = q_a, \tag{I$_1$}$$

$$T - c\mathbf{as} = E. \tag{I$_2$}$$

q_a being the vertical component of the moment of momentum and E the total energy. Both of these functions of the state of the system conserve constant values during its motion.

Other invariants, or integrals, for a rotating heavy rigid system, *in the general case*, *i.e.* for any principal moments K_1, K_2, K_3, no one has yet succeeded in finding. Observe that in the absence of impressed forces the *whole* vector $\mathbf{q}$ has been constant, while, in the present case, only its vertical component is constant. Instead of three we have now only one scalar invariant, that is, together with the energy, two scalar invariants, whereas before we had four. Two, due to the force of gravity, are gone, or at least have hidden themselves so deeply that mathematicians cannot find them. In fact, the general problem of a rotating heavy rigid system is still awaiting its solution, *i.e.* a reduction to quadratures.

A third invariant necessary, in addition to (I$_1$) and (I$_2$), for the solution of the problem, has been discovered only in a few *special* cases, viz. under special suppositions as to the moments of inertia and to the position of the centre of gravity of the system.

The most remarkable is the special case associated with the names of Lagrange and Poisson, viz. in which *the centre of gravity C lies on one of the principal axes* (passing through O), say $\mathbf{s} = \mathbf{x}_1$, while the moments of inertia corresponding to the other two axes are *equal*, *i.e.* $K_2 = K_3$. The ellipsoid of inertia becomes in this case an ellipsoid of rotation, $\mathbf{x}_1$ *being its axis of symmetry*.

A third integral has also been found, in some other special cases, by Hess, Mrs. Kowalewski and Tschapligin.*

Here we shall limit ourselves to the consideration of the **case of Lagrange and Poisson**, which may be defined, shortly, thus:

$$\mathbf{s} = \mathbf{x}_1, \quad K_2 = K_3. \tag{43}$$

*See Routh's *Advanced Rigid Dynamics*, or P. Appell's *Traité de mécanique rationelle*, Vol. II. Paris, 1904.

Multiply (40′), scalarly, by $\mathbf{s}=\mathbf{s}'$ and remember that $\mathbf{s}\mathrm{V}\mathbf{s}\mathbf{a}=0$, identically; then

$$\mathbf{s}'d\mathbf{q}'/dt=\mathbf{s}\mathrm{V}\mathbf{q}\mathbf{p};$$

but $d\mathbf{s}'/dt=0$; hence

$$\frac{d}{dt}(\mathbf{q}'\mathbf{s}')=\frac{d}{dt}(\mathbf{q}\mathbf{s})=\mathbf{s}\mathrm{V}\mathbf{q}\mathbf{p}. \tag{44}$$

In this equation, which is *generally* true, we could write $\mathbf{q}\mathbf{s}$ instead of $\mathbf{q}'\mathbf{s}'$ after the symbol d/dt, since the scalar product of two vectors is certainly independent, also as regards its rate of variation, of any system of reference.

Now, in the special case considered, the product $\mathbf{s}\mathrm{V}\mathbf{q}\mathbf{p}$ on the right side of (44) vanishes. To see this, remember that, by (43), the ellipsoid of inertia is an ellipsoid of rotation, so that the vectors $\mathbf{p}$, $\mathbf{q}$ are in the plane passing through the axis of symmetry $\mathbf{x}_1$; hence

$$\mathbf{s}\mathrm{V}\mathbf{q}\mathbf{p}=\mathbf{x}_1\mathrm{V}\mathbf{q}\mathbf{p}=0,$$

since $\mathbf{x}_1$, $\mathbf{p}$, $\mathbf{q}$ are coplanar.

Thus, by (44), $$\frac{d}{dt}(\mathbf{q}\mathbf{x}_1)=\frac{dq_1}{dt}=0$$

or $q_1=K_1p_1=const.$, whence also $p_1=const.$

The third invariant, in the case of Lagrange and Poisson, is thus

$$\mathbf{p}\mathbf{x}_1=p_1, \tag{I_3}$$

i.e. the angular velocity of the system about its axis of symmetry. In other words, the component of the angular velocity taken along the axis of symmetry retains its initial value.

Combining (I_3) with the two other, general, invariants (I_1), (I_2), the complete solution may be reached at once, that is to say, the angle $\theta=\sphericalangle\mathbf{a},\mathbf{s}$ (Fig. 31) or more conveniently $\cos\theta=\mathbf{a}\mathbf{s}$ and, say, the two remaining 'angles of Euler' ϕ, ψ may be expressed for every value of t by elliptic functions; whence, by ordinary differentiation, p_2, p_3, and thereby also the resultant angular velocity p, will follow.* Instead of Euler's other parameters may also be easily introduced. This is left to the reader, for his own exercise, since we only wished to deduce here the third invariant, corresponding to the Lagrange-Poisson case, in the shortest vector way.

The differential equation of motion of a (compound or) **physical pendulum** in its most usual form, *i.e.* of a heavy rigid body which can

*See, for instance, Appell, *loc. cit.* p. 189, etc. For the definition of the usual angles 'θ, ϕ, ψ' see the same work or Routh's *Rigid Dynamics*.

turn only about a fixed horizontal axis, having thus *one* degree of freedom only, may be obtained immediately from the invariant

$$T-c\mathbf{a}\mathbf{s}=E=\text{const.}$$

This simple system, the state of which is determined completely by *two* scalars, namely by the angle $\theta=\sphericalangle\mathbf{a},\mathbf{s}$ and by the angular velocity $\dot\theta$, has but $2-1=one$ essential invariant, and this is the energy E,—or any function of E alone, which is exactly the same thing.

Denoting the pendulum's moment of inertia about the fixed horizontal axis by B and remembering that $\mathbf{a}\mathbf{s}=\cos\theta$, we have

$$E=\tfrac{1}{2}B\dot\theta^2-c.\cos\theta=\text{const.},$$

whence, by differentiation,

$$B\ddot\theta=-c.\sin\theta, \tag{45}$$

a well-known result. The coefficient c is, by (39), equal to MgS or to the so-called **directive moment** of the pendulum.

Kinematical Relations.

To determine the instantaneous position of the rotating rigid system relative to a 'fixed' system of reference, the well-known *angles of Euler* θ, ψ, ϕ are usually employed. The component angular velocities p_1, p_2, p_3 are then expressed by linear functions of the fluxes $\dot\theta$, $\dot\psi$, $\dot\phi$ with coefficients depending on simple trigonometrical functions of the angles θ, ψ, ϕ themselves (cf. one of the works cited above). These relations are known under the name of *Euler's kinematical equations.*

Euler's angles are very convenient parameters, especially as their number is *three*, *i.e.* equal to the number of degrees of freedom of a rotating rigid system. Nevertheless their choice is based on distinguishing one of the three axes from the other two; in other words, the angles θ, ψ, ϕ are not *co*ordinated with one another (but two are *sub*ordinated), as regards their geometrical meaning; whereby the symmetry of the formulae is impaired.

In the vector treatment of this subject the kinematical relations are, in fact, contained already in the equation (22), when applied to any vector $\mathbf{w}$ *which is invariably attached to the given rigid system** (*i.e* when $d\mathbf{w}'/dt=0$), that is to say, in the equation

$$\dot{\mathbf{w}}=\frac{d\mathbf{w}}{dt}=\mathrm{V}\mathbf{p}\mathbf{w}.$$

*With a given fixed point O of the system, always tacitly assumed.

Now, such vectors are for instance the three unit vectors $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ representing the rigid system's principal axes, these certainly being attached to the system. Hence, we can put, in the last equation, $\mathbf{w}=\mathbf{x}_1$, or $\mathbf{x}_2$, or $\mathbf{x}_3$, and thus we shall obtain the three vectorial differential equations

$$\left.\begin{aligned}\dot{\mathbf{x}}_1&=\mathrm{V}\mathbf{p}\mathbf{x}_1\\ \dot{\mathbf{x}}_2&=\mathrm{V}\mathbf{p}\mathbf{x}_2\\ \dot{\mathbf{x}}_3&=\mathrm{V}\mathbf{p}\mathbf{x}_3\end{aligned}\right\}, \tag{46}$$

which can be regarded as the expression of **the kinematical relations.** In this system of equations, the *rôle* of all the three principal axes is the same. It is true that we have here three unit vectors $\mathbf{x}_1$, etc., amounting to *six* scalars, whereas the rotating rigid system has only *three* degrees of freedom; in other words, $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ are not mutually independent variables, as θ, ψ, ϕ are, but, on the other hand, by using them and the form (46), none of the moving axes is specially distinguished, so that the formulae show a perfect symmetry.

Remember that $\mathbf{x}_1$, etc., are not only *unit* vectors, but also mutually perpendicular; we have not only

$$x_1^2=x_2^2=x_3^2=1,$$

but also $$\mathbf{x}_2\mathbf{x}_3=\mathbf{x}_3\mathbf{x}_1=\mathbf{x}_1\mathbf{x}_2=0,$$

that is, we have in reality only *three* independent scalar parameters, *i.e.* as many as are the degrees of freedom of the system in question. We may, however, retain $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ in the kinematical relations, *keeping in mind* the last scalar conditions, but not using them explicitly for the process of elimination. This method of treatment will secure symmetry, and thus, for *general* considerations, will be more convenient than scalar parameters of which the number is explicitly reduced to three.

In the kinematical relations (46) the fluxes $\dot{\mathbf{x}}_1$, etc., are expressed by the angular velocity $\mathbf{p}$ of the system and by the $\mathbf{x}_1$, etc., themselves. Conversely, we may also express at once the components of $\mathbf{p}$ along the moving principal axes by $\dot{\mathbf{x}}_1$, etc., and by $\mathbf{x}_1$, etc. In fact, multiplying the last of equations (46), scalarly, by $\mathbf{x}_2$, we get

$$\mathbf{x}_2\dot{\mathbf{x}}_3=\mathbf{x}_2\mathrm{V}\mathbf{p}\mathbf{x}_3=\mathbf{p}\mathrm{V}\mathbf{x}_3\mathbf{x}_2;$$

but $$\mathrm{V}\mathbf{x}_3\mathbf{x}_2=-\mathrm{V}\mathbf{x}_2\mathbf{x}_3=-\mathbf{x}_1;\quad \mathbf{p}\mathbf{x}_1=p_1;$$

$$\therefore\quad -\mathbf{x}_2\dot{\mathbf{x}}_3=p_1.$$

Now, differentiating $\mathbf{x}_2\mathbf{x}_3=0$, we have $\mathbf{x}_2\dot{\mathbf{x}}_3=-\dot{\mathbf{x}}_2\mathbf{x}_3$, whence $p_1=\dot{\mathbf{x}}_2\mathbf{x}_3$. Two other similar equations may be obtained immediately by cyclic permutation of the indices 1, 2, 3. Thus, the required relations will assume the form

$$\left.\begin{aligned}p_1&=\dot{\mathbf{x}}_2\mathbf{x}_3\\ p_2&=\dot{\mathbf{x}}_3\mathbf{x}_1\\ p_3&=\dot{\mathbf{x}}_1\mathbf{x}_2\end{aligned}\right\}, \tag{47}$$

in which none of the axes is distinguished.

I do not say that our $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ will also for *special* purposes of calculation be equally convenient with the parameters θ, ψ, ϕ. They are, however, most convenient for the general treatment of the subject as here considered.

The Line-Integral and the Curl of a Vector illustrated by the Example of a Moving Rigid Body.

Let us imagine any *circuit* or closed curve s drawn in a moving rigid body; let $d\mathbf{s}$ be one of its elements, as regards length and direction. Denote again by $\mathbf{v}$ the resultant velocity of any point of the body, relatively to O, and consider the line-integral of $\mathbf{v}$ taken round the whole circuit s,

$$I=\int\mathbf{v}\,d\mathbf{s}.$$

If the rigid body rotates about its fixed point $O=O$, we have $\mathbf{v}=\mathrm{V}\mathbf{p}\mathbf{r}$, $\mathbf{p}$ being the instantaneous angular velocity of the body; but in the more general case the body, as a whole, may have also a velocity of *translation*, say $\mathbf{v}_0$; then

$$\mathbf{v}=\mathbf{v}_0+\mathrm{V}\mathbf{p}\mathbf{r},$$

where the vector $\mathbf{v}_0$, as also $\mathbf{p}$, has equal absolute values and directions for all the points of the rigid body, *i.e.* is independent of $\mathbf{r}$, at any instant whatever. In other words, $\mathbf{p}$ and $\mathbf{v}_0$ belong to the whole of the rigid body, whereas the $\mathbf{r}$'s distinguish the individual points of the body.

Now, $$\mathbf{v}\,d\mathbf{s}=\mathbf{v}_0\,d\mathbf{s}+d\mathbf{s}.\mathrm{V}\mathbf{p}\mathbf{r}=\mathbf{v}_0\,d\mathbf{s}+\mathbf{p}\mathrm{V}\mathbf{r}\,d\mathbf{s};$$

but $\int\mathbf{v}_0\,d\mathbf{s}=\mathbf{v}_0\int d\mathbf{s}=0$, the elementary vectors $d\mathbf{s}$ constituting a *closed* chain. Moreover, $\mathbf{p}$ being the same for all points, it may be written before the symbol of integration; thus

$$I=\mathbf{p}\mathbf{W},\ \text{where}\ \mathbf{W}=\int\mathrm{V}\mathbf{r}\,d\mathbf{s}$$

or $$I=p\,W_p,$$

W_p being the component of the vector $\mathbf{W}$ taken along the instantaneous axis of rotation.

Now, the tensor or the absolute value of $\frac{1}{2}\mathrm{V}\mathbf{r}\,d\mathbf{s}$ is the area of the triangle O, $\mathbf{r}$, $d\mathbf{s}$ (Fig. 32); hence

$$\tfrac{1}{2}W_p$$

is the area σ_p of the plane figure bounded by the projection of the

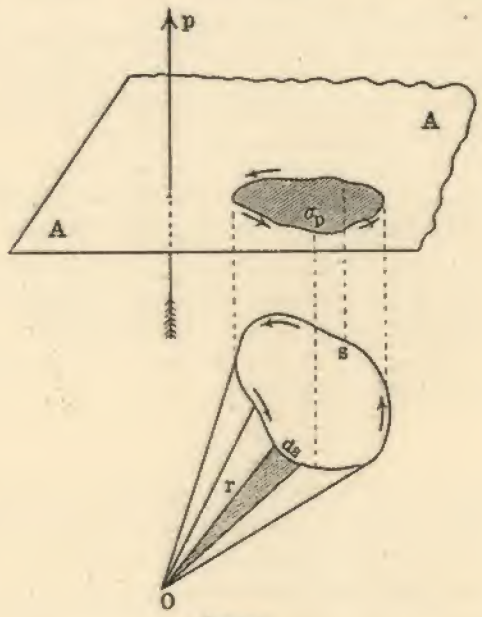

Fig. 32.

circuit s on the plane AA normal to $\mathbf{p}$. Thus, $I = 2p\sigma_p$, or

$$\frac{I}{\sigma_p} = \frac{\int \mathbf{v}\,d\mathbf{s}}{\sigma_p} = 2p.$$

If the circuit s be a plane curve, drawn in a plane normal to $\mathbf{p}$, we should have simply

$$\frac{I}{\sigma} = 2p,$$

where σ is the area of the plane figure bounded by s itself. The circuit s may also be contracted indefinitely, without any change in the above relation.

Hence, remembering the definition of the operator *curl*, it is seen from the preceding that, in the case considered,

$$\operatorname{curl}\mathbf{v} = 2\mathbf{p},$$

i.e. the *curl* of the resultant velocity of the points of the rigid body is equal to twice the *angular velocity* of the body, both in absolute value and direction.

In the next chapter we shall see that, when $\mathbf{v}$ is the velocity in a deformable body (or medium), $\operatorname{curl}\mathbf{v}$ is the double *vortex velocity* of a particle. Whence the name of *rotation* used commonly as a synonym of *curl*. Both '*curl*' or '*quirl*' and '*rotation*' were proposed first by Maxwell (*Electricity and Magnetism*, Vol. I., and *Scientific Papers*); H. A. Lorentz and nearly all continental authors write *rot* (= *rotation*); Heaviside and most English vector-writers use the symbol *curl*, which also will be used in this book.

The operator *curl* is (by Chapter I.) identical with the Hamiltonian operator ∇ applied *vectorially*, *i.e.*

$$\operatorname{curl}\mathbf{w} = \mathrm{V}\nabla\mathbf{w}$$

for any vector $\mathbf{w}$; hence, if, for example, the ordinary components w_1, w_2, w_3 are used, taken along the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ constituting a normal *right-handed* system,

$$\operatorname{curl}\mathbf{w} = \mathbf{i}\left(\frac{\partial w_3}{\partial y} - \frac{\partial w_2}{\partial z}\right) + \mathbf{j}\left(\frac{\partial w_1}{\partial z} - \frac{\partial w_3}{\partial x}\right) + \mathbf{k}\left(\frac{\partial w_2}{\partial x} - \frac{\partial w_1}{\partial y}\right),$$

x, y, z being the usual Cartesian coordinates along $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$,—or, in determinant form,

$$\operatorname{curl}\mathbf{w} = \mathrm{V}\nabla\mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \nabla_1 & \nabla_2 & \nabla_3 \\ w_1 & w_2 & w_3 \end{vmatrix}, \qquad (\text{XI.})$$

where ∇_1, etc., are the 'components' of ∇, *i.e.* $\nabla_1 = \dfrac{\partial}{\partial x}$, etc.

Writing $\mathbf{w} = \mathbf{v}_0 + \mathrm{V}\mathbf{p}\mathbf{r} = \mathbf{v}$ and using (XI.), the reader will easily verify the above relation $\operatorname{curl}\mathbf{v} = 2\mathbf{p}$, if he remembers that $\mathbf{v}_0$, $\mathbf{p}$ are constants.

CHAPTER V.

GENERAL MECHANICS OF DEFORMABLE BODIES.

General Remarks.

A MOST important part in the treatment of problems to be now considered is played by the linear vector operator of which we learned a little in the last chapter, when considering the dynamical properties of a rigid system. But there we used only the *symmetrical* operator, of which the principal axes were mutually perpendicular, whereas for non-rigid bodies this operator does not suffice; here, in the most essential problems, we shall meet with the *general* linear vector operator, *i.e.* generally **non-symmetrical operator.** And, to avoid mathematical digressions in the course of development of the mechanics of the subject, we shall in the first place put together, in the next section, the most fundamental properties of this operator, the discovery of which, after Hamilton, is principally due to J. Willard Gibbs.* A sufficiently complete exposition of its properties has been given by Heaviside in his excellent book on Electromagnetism.†

Devoting to this purely mathematical subject a whole (but not very extensive) section, we shall simplify and shorten considerably the treatment of what has to come later, especially of the deformations or strains of a body.

Fundamental Properties of the Linear Vector Operator.

A vector **B** is said to be a **linear vector function** of another vector **A** when its components B_1, B_2, B_3 along any (arbitrary) three non-

* J. W. Gibbs, *Vector Analysis* ('not published'), Newhaven, 1881-4; Wilson, *Vector Analysis, etc., founded upon the lectures of J. W. Gibbs*, New York and London, 1902.

† Heaviside, *loc. cit.* §§ 168-173.

coplanar axes are linear functions of the components A_1, A_2, A_3 of the vector **A**. If this relation be satisfied for some given system of such axes, it is easily proved to hold also for any other system of axes.*

As such a system of reference, the right-handed system of unit vectors **i**, **j**, **k** may, for instance, be taken, so that $\mathbf{i}^2 = 1$, etc., $\mathbf{ij} = 0$, etc., $\mathrm{V}\mathbf{ij} = \mathbf{k}$, etc.

The symbol of the operations to be performed on **A** in order to get **B** is called **the linear vector operator.** We shall denote it, generally, by ω, thus writing

$$\mathbf{B} = \omega\mathbf{A}. \qquad \text{(XII.)}$$

This will, by definition, be equivalent to the following three scalar equations with nine scalar coefficients ω_{11}, ω_{12}, etc.:

$$\left.\begin{aligned} B_1 &= \omega_{11}A_1 + \omega_{12}A_2 + \omega_{13}A_3 \\ B_2 &= \omega_{21}A_1 + \omega_{22}A_2 + \omega_{23}A_3 \\ B_3 &= \omega_{31}A_1 + \omega_{32}A_2 + \omega_{33}A_3 \end{aligned}\right\}. \qquad \text{(XII.}a\text{)}$$

Thus, the **general** linear vector operator is characterised by **nine** mutually independent scalar data; such data, as the above ω_{11}, ω_{12}, ... ω_{33}, may be replaced by other scalars, but their number will not thereby be changed. Instead of nine scalar data three vectors may be substituted; $3 \times 3 = 9$.

The formula (XII.) is a short expression for the three scalar equations (XII.*a*). Omitting the vector **A**, to be operated on, we may conveniently write for the operator:

$$\omega = \begin{vmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{vmatrix} \qquad \text{(XII.}b\text{)}$$

In order to get from this the vector $\mathbf{B} = \omega\mathbf{A}$, we have only to write A_1 as a factor in the whole first column, A_2 in the second, A_3 in the third column; then, the first horizontal line (its terms

* The above definition of a linear vector function may also be deduced from another, in some regards more convenient and less artificial definition, viz. as stated by Gibbs-Wilson (*loc. cit.* p. 262):

'A continuous vector function of a vector is said to be a *linear* vector function when the function of the sum of any two vectors is the sum of the functions of those vectors. That is, the function f is linear if

$$f(\mathbf{r}_1 + \mathbf{r}_2) = f(\mathbf{r}_1) + f(\mathbf{r}_2).'$$

being joined by the signs +) gives B_1, the second B_2, the third B_3. Now, it is not necessary to write the 'A' nine times: it is sufficient to imagine it done. With this understanding the scheme or symbol (XII.*b*) will be very useful.

Generally, $\omega_{23} \neq \omega_{32}$, etc. But if it happens that

$$\omega_{23} = \omega_{32}, \quad \omega_{31} = \omega_{13}, \quad \omega_{12} = \omega_{21},$$

then ω is called a **symmetrical linear operator**, and the vector $\mathbf{B} = \omega\mathbf{A}$ is called a **symmetrical linear function** of $\mathbf{A}$.

The *symmetrical* operator will henceforth be denoted by a capital omega (Ω). Thus, the operator Ω will be completely characterised by $9 - 3 = six$ mutually independent scalars, say by the six coefficients Ω_{11}, Ω_{22}, Ω_{33}, Ω_{23}, Ω_{31}, Ω_{12}. It may be written

$$\Omega = \begin{vmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} \\ & \Omega_{22} & \Omega_{23} \\ & & \Omega_{33} \end{vmatrix}$$

The diagonal, joining $_{11}$ and $_{33}$ is an axis of symmetry of the whole square table; remembering this, it is unnecessary to fill out the vacant places.

By permutation of the indices in the general operator ω, *i.e.* by writing ω_{32} instead of ω_{23}, etc., another linear operator ω' is obtained, which is said to be **conjugate** to ω. This relation is, of course, reciprocal, *i.e.* ω is also conjugate to ω'.

Now, by the definition of the symmetrical operator, $\Omega_{32} = \Omega_{23}$, etc.; thus, Ω is a **self-conjugate** operator.

Let ϕ and ψ denote any two linear vector operators; suppose that

$$\mathbf{B} = \phi\mathbf{A} + \psi\mathbf{A};$$

then we may write also $\mathbf{B} = \omega\mathbf{A}$,

ω being the linear operator, the coefficients of which are equal to the sums of the corresponding coefficients of ϕ, ψ, *i.e.*

$$\omega_{11} = \phi_{11} + \psi_{11}, \quad \omega_{12} = \phi_{12} + \psi_{12}, \quad \text{etc.}$$

This may be expressed by writing

$$\omega = \phi + \psi,$$

and by calling ω the *sum* of the operators ϕ, ψ. It follows at once that

$$\psi + \phi = \phi + \psi.$$

The same holds for subtraction. Whence follow also the rules for the multiplication or the division of a linear vector operator by a scalar number. If n be an arbitrary scalar, then $n\omega$ is simply a linear operator which has the coefficients $n\omega_{11}$, $n\omega_{12}$, etc.; hence we may write also ωn instead of $n\omega$.

These simple properties are sufficient to prove one of the most fundamental theorems, viz. that the *general* linear vector function $\omega\mathbf{A}$ may always be split up into a *symmetrical* linear function of $\mathbf{A}$ and a *vector product* of $\mathbf{A}$ by another vector $\mathbf{c}$ which is characteristic for the operator ω, *i.e.*

$$\omega\mathbf{A} = \Omega\mathbf{A} + \mathrm{V}\mathbf{c}\mathbf{A},$$

or, omitting the vector $\mathbf{A}$ to be operated on,

$$\omega = \Omega + \mathrm{V}\mathbf{c}.$$

At the same time, the symmetrical operator Ω and the vector $\mathbf{c}$ may be expressed without any ambiguity by the properties of the operator ω.

To see this, write for the given general operator ω, instead of (XII.*b*):

$$\omega = \begin{vmatrix} \omega_{11} & \frac{\omega_{12}+\omega_{21}}{2} & \frac{\omega_{13}+\omega_{31}}{2} \\ \frac{\omega_{21}+\omega_{12}}{2} & \omega_{22} & \frac{\omega_{23}+\omega_{32}}{2} \\ \frac{\omega_{31}+\omega_{13}}{2} & \frac{\omega_{32}+\omega_{23}}{2} & \omega_{33} \end{vmatrix} + \tfrac{1}{2} \begin{vmatrix} 0 & \omega_{12}-\omega_{21} & \omega_{13}-\omega_{31} \\ \omega_{21}-\omega_{12} & 0 & \omega_{23}-\omega_{32} \\ \omega_{31}-\omega_{13} & \omega_{32}-\omega_{23} & 0 \end{vmatrix}$$

which is legitimate, according to the preceding remarks on addition. Now, the first operator on the right side is already a symmetrical operator Ω, of which the coefficients are

$$\Omega_{\iota\kappa} = \tfrac{1}{2}(\omega_{\iota\kappa} + \omega_{\kappa\iota}) = \Omega_{\kappa\iota}; \quad \iota, \kappa = 1, 2, 3.$$

Remembering the definition of the conjugate operator ω', we may write $\Omega = \tfrac{1}{2}(\omega + \omega')$.

Hence, in order to prove the theorem enunciated, we have only to transform the second operator. Denote it for the moment by ψ, so that $\omega = \Omega + \tfrac{1}{2}\psi$.

Then, operating on the vector $\mathbf{A} = \mathbf{i}A_1 + \mathbf{j}A_2 + \mathbf{k}A_3$, we have

$$\begin{aligned} \psi\mathbf{A} = \mathbf{i}[0 + (\omega_{13} - \omega_{31})A_3 - (\omega_{21} - \omega_{12})A_2] \\ + \mathbf{j}[(\omega_{21} - \omega_{12})A_1 - (\omega_{32} - \omega_{23})A_3] \\ + \mathbf{k}[(\omega_{32} - \omega_{23})A_2 - (\omega_{13} - \omega_{31})A_1]; \end{aligned}$$

but the sum of all the terms on the right side is the vector product of the vector

$$\mathbf{c} = \mathbf{i}(\omega_{32} - \omega_{23}) + \mathbf{j}(\omega_{13} - \omega_{31}) + \mathbf{k}(\omega_{21} - \omega_{12})$$

by the vector $\mathbf{A}$, *i.e.*

$$\psi\mathbf{A} = \mathrm{V}\mathbf{c}\mathbf{A},$$

or, omitting again the vector operated on,

$$\psi = \mathrm{V}\mathbf{c},$$

which completely proves the theorem.

Thus, we have, for any linear vector operator:

$$\left.\begin{aligned} \omega &= \Omega + \tfrac{1}{2}\mathrm{V}\mathbf{c}, \text{ where} \\ \Omega &= \tfrac{1}{2}(\omega + \omega'), \\ \mathbf{c} &= \mathbf{i}(\omega_{32} - \omega_{23}) + \mathbf{j}(\omega_{13} - \omega_{31}) + \mathbf{k}(\omega_{21} - \omega_{12}). \end{aligned}\right\} \quad \text{(XIII.)}$$

Whence it is seen also that the decomposition of the general operator into a symmetrical part and a non-symmetrical part (the last being the so-called **antisymmetrical** part) can be effected in but one way, both Ω and $\mathbf{c}$ being completely determined by the properties of the given operator ω.

Conversely, the symmetrical operator Ω together with the vector $\mathbf{c}$ determine completely the general operator ω. Now, Ω implies *six* scalar data, $\mathbf{c}$ *three*, making together *nine*, as we know it must be.

Let us now apply the operator ω to a bundle of vectors $\mathbf{A}$ having the same origin and the same tensor A, but having all possible different directions in space. Then the symmetrical part Ω of ω will transform the sphere $A = const.$ into an *ellipsoid* of which the axes will coincide with the principal axes of Ω, and the remaining part $\frac{1}{2}\mathrm{V}\mathbf{c}$, if c be supposed *infinitesimal*, will *turn* this ellipsoid as a whole through the angle $\frac{1}{2}c$ round the axis determined by the direction of $\mathbf{c}$.

The above theorem (XIII.), which, in fact, has been the chief object of the present mathematical digression, will be found to render very important service in considering strains and allied matter.

A few remarks may further be added here which will be of some use in what will follow.

Following Gibbs' example, we may determine the general linear operator ω by three mutually independent vectors, say $\mathbf{O}_1$ etc.

$$\mathbf{O}_1 = \mathbf{i}\omega_{11} + \mathbf{j}\omega_{12} + \mathbf{k}\omega_{13},$$
$$\mathbf{O}_2 = \mathbf{i}\omega_{21} + \mathbf{j}\omega_{22} + \mathbf{k}\omega_{23},$$
$$\mathbf{O}_3 = \mathbf{i}\omega_{31} + \mathbf{j}\omega_{32} + \mathbf{k}\omega_{33}.$$

Then, by (XII.*a*), $B_1 = \mathbf{O}_1\mathbf{A}$, $B_2 = \mathbf{O}_2\mathbf{A}$, $B_3 = \mathbf{O}_3\mathbf{A}$

or

$$\mathbf{B} = \omega\mathbf{A} = \mathbf{i}.\mathbf{O}_1\mathbf{A} + \mathbf{j}.\mathbf{O}_2\mathbf{A} + \mathbf{k}.\mathbf{O}_3\mathbf{A};$$

hence, the operator ω will assume the form of the so-called **dyadic**:

$$\omega = \mathbf{i}.\mathbf{O}_1 + \mathbf{j}.\mathbf{O}_2 + \mathbf{k}.\mathbf{O}_3. \quad \text{(XIV.)}$$

The dots after $\mathbf{i}$, etc., are **separators**,* reminding us that the vector $\mathbf{A}$ (to be operated on by ω) has first of all to be multiplied scalarly by $\mathbf{O}_1$, or $\mathbf{O}_2$, or $\mathbf{O}_3$, and that the scalars thus obtained are to be multiplied, respectively, by $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, and added.

As to the symmetrical operator Ω, we know already that it has, generally, three mutually perpendicular principal axes to which correspond the three principal values, say Ω_1, Ω_2, Ω_3 (ordinary scalars). For this operator, it is most convenient to take the orthogonal system of reference $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ along the principal axes themselves; then

$$\Omega_{11} = \Omega_1, \quad \Omega_{12} = 0, \text{ etc.},$$

or in the tabular form:

$$\Omega = \begin{vmatrix} \Omega_1 & 0 & 0 \\ 0 & \Omega_2 & 0 \\ 0 & 0 & \Omega_3 \end{vmatrix}$$

or, finally,

$$\mathbf{B} = \Omega\mathbf{A} = \mathbf{i}\Omega_1 A_1 + \mathbf{j}\Omega_2 A_2 + \mathbf{k}\Omega_3 A_3, \quad \text{(XV.)}$$

as we wrote previously, when we were considering the dynamics of a rigid system. (The 'K' has, of course, been a special case of Ω.) We then saw also that, for any pair of vectors $\mathbf{A}$, $\mathbf{C}$,

$$\mathbf{C}\Omega\mathbf{A} = \mathbf{A}\Omega\mathbf{C}, \quad \text{(XVI.)}$$

i.e. the scalar product of a vector by a symmetrical linear function of another vector is equal to the scalar product of the second by the same function of the first.

Observe that this property *does not* belong to a *non-symmetrical* operator ω; in fact, as may easily be shown, we have in this case the more general relation

$$\mathbf{C}\omega\mathbf{A} = \mathbf{A}\omega'\mathbf{C}, \quad \text{(XVI'.)}$$

ω' being the conjugate of ω. The relation (XVI.) follows from (XVI'.) as its special case; since Ω is *self*-conjugate, *i.e.* $\Omega' = \Omega$.

* In Heaviside's denotation, whereas Gibbs uses the dots as symbols of scalar products (and little crosses as symbols of vector products).

Strains.

Let the vector $\mathbf{a}$ define (relatively to the system of reference O) the position of an individual point or 'particle' of a body or medium *before the deformation*, and the vector $\mathbf{r}$ the position of the same particle *after the deformation*, or, if we prefer it: the respective positions of the same material particle at the initial instant t_0 and at any other instant t.

The individual particle, the position of which before the deformation is determined by $\mathbf{a}$, we shall call *the particle* $\mathbf{a}$, or *the point* $\mathbf{a}$.

Fig. 33 represents the position of such a point, 1 before, and 1′ after the deformation. Similarly, 2, 3 are the positions of two other points before, and 2′, 3′ after the deformation.

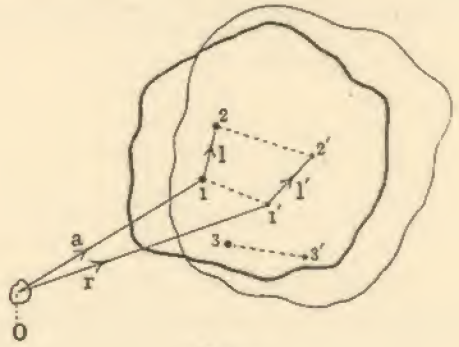

FIG. 33.

We shall assume that the vector $\mathbf{r}$ is a *continuous* function of the vector $\mathbf{a}$, admitting definite and continuous derivatives with respect to any component of $\mathbf{a}$. In speaking of the continuity of a vector, we mean, of course, continuity as regards both tensor and direction.

Discontinuities will be considered later on.

Let 1 and 2 (Fig. 33) represent the positions of two points at an infinitesimal distance from one another,

$$\mathbf{a}=\mathbf{a}_1 \quad \text{and} \quad \mathbf{a}=\mathbf{a}_2,$$

before the deformation of the body. The infinitesimal vector

$$\mathbf{l}=\mathbf{a}_2-\mathbf{a}_1,$$

drawn from 1 towards 2, will characterise some *linear element* or one-dimensional assemblage of individual points of the body.

The positions 1′, 2′ of these points *after the deformation* will be denoted by

$$\mathbf{r}=\mathbf{r}_1 \quad \text{and} \quad \mathbf{r}=\mathbf{r}_2$$

respectively, so that

$$\mathbf{l}'=\mathbf{r}_2-\mathbf{r}_1$$

is what $\mathbf{l}$ becomes after the deformation, namely the vector 1′→2′ (Fig. 33).

The infinitesimal vectors $\mathbf{l}$ and $\mathbf{l}'$ represent the length and the direction of one and the same linear element before and after the deformation, respectively.

What, then, becomes of the element $\mathbf{l}$ after the deformation? In other words: What is the relation between $\mathbf{l}'$ and $\mathbf{l}$?

Let us denote by ∇ the Hamiltonian in 'the space $\mathbf{a}$,' *i.e.* in that space in which the position of a point is given by $\mathbf{a}$.

Then the change of any magnitude ϕ (scalar or vector) due to passing from the point $\mathbf{a}$ to the point $\mathbf{a}+\mathbf{l}$ will be expressed by

$$l\frac{\partial\phi}{\partial l}=(\mathbf{l}\nabla)\phi,$$

infinitesimals of higher order than l being neglected.* The operator $\mathbf{l}\nabla$ has (by Chapter I.) *a scalar character*, expressing simply the component of the gradient or slope in the direction of $\mathbf{l}$, multiplied by the length l of this line-element.

This can be applied immediately to the vector $\mathbf{r}$, *i.e.* to $\phi=\mathbf{r}$. Thus we get the required expression for $\mathbf{r}_2-\mathbf{r}_1$ or $\mathbf{l}'$, namely

$$\mathbf{l}'=(\mathbf{l}\nabla)\mathbf{r}. \tag{48}$$

If $\mathbf{r}$ be given as a function of $\mathbf{a}$, *i.e.* if we know precisely the deformation to be dealt with, then the operation indicated on the right side of (48) may be performed, for any $\mathbf{a}$ and for any $\mathbf{l}$, *i.e.* for any line-element having its origin in any point of the body. Thus, (48) gives the length and the direction of any linear element after the deformation in terms of its length and direction before the deformation.

The direction of the vector $(\mathbf{l}\nabla)\mathbf{r}$ is, of course, generally different from that of $\mathbf{l}$. The element $\mathbf{l}$ will, in other words, change not only in length l but also in direction. In addition, it will be also displaced as a whole, this purely translational displacement being namely 1→1′ or $\mathbf{r}-\mathbf{a}$ (Fig. 33).

* If it should happen that $(\mathbf{l}\nabla)\phi=0$, then, of course, infinitesimals of the second, and eventually of higher orders, have to be considered.

The general term 'deformation' implies all these changes. The proper deformation of a linear element will be only its elongation (contraction = negative elongation); the deformation of an element of surface or of volume will imply change both of dimensions and shape of the elementary area or volume. But all the changes of two- or three-dimensional elements may be easily deduced from the changes of linear or one-dimensional elements. Therefore the formula (48), or its scalar equivalents, has a most fundamental importance in the whole theory of deformations or strains.

Instead of the vector $\mathbf{r}$, determining the position of the point $\mathbf{a}$ after deformation, the **displacement** of this point, *i.e.* the vector

$$\mathbf{D} = \mathbf{r} - \mathbf{a}, \tag{49}$$

may easily (and conveniently) be substituted in the fundamental formula and in those which will follow from it.

In fact, $l\nabla$ being a *distributive* operator, we have

$$l' = (l\nabla)(\mathbf{D} + \mathbf{a}) = (l\nabla)\mathbf{D} + (l\nabla)\mathbf{a};$$

now, the second term on the right side is the change of $\mathbf{a}$ itself due to passing from the point $\mathbf{a}_1$ to the point $\mathbf{a}_2$, *i.e.* simply $(l\nabla)\mathbf{a} = \mathbf{a}_2 - \mathbf{a}_1 = l$; hence

$$l' = l + (l\nabla)\mathbf{D}. \tag{48'}$$

But in what follows the preceding formula (48) will be used, as being somewhat simpler. In the results, when once obtained, it will always be very easy to pass from $\mathbf{r}$ to the displacement $\mathbf{D}$ by means of the simple relation (49).

By (48) the vector l' is already seen to be a linear function of l; hence it may be written

$$l' = \omega l.$$

But we have still to find the properties which in this case belong to the linear vector operator ω, *i.e.* to split ω into its symmetrical and non-symmetrical parts, to determine the precise form of each of these parts as dependent on the given deformation and, finally, to give their physical interpretation.

By (XIII.) we may write at any rate:

$$l' = \Omega l + \tfrac{1}{2} V c l; \tag{50}$$

thus, the problem is reduced to developing the expressions for the symmetrical operator Ω and for the vector $\mathbf{c}$.

Both must be deduced, of course, from the particular linear relation (48):

$$l' = (l\nabla)\mathbf{r}. \tag{48 *bis*}$$

In order to avail ourselves of the reasoning and the symbols of the last section, let us employ, for the moment, as a system of reference, the normal unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, *i.e.* let us write

$$\mathbf{r} = \mathbf{i}r_1 + \mathbf{j}r_2 + \mathbf{k}r_3,$$

and similarly

$$l = \mathbf{i}l_1 + \mathbf{j}l_2 + \mathbf{k}l_3,$$
$$l' = \mathbf{i}l_1' + \mathbf{j}l_2' + \mathbf{k}l_3',$$
$$\nabla = \mathbf{i}\nabla_1 + \mathbf{j}\nabla_2 + \mathbf{k}\nabla_3,$$

∇_1, etc., being the 'components' of ∇, *i.e.* the derivators in the space $\mathbf{a}$ or

$$\nabla_1 = \frac{\partial}{\partial a_1}, \quad \nabla_2 = \frac{\partial}{\partial a_2}, \quad \nabla_3 = \frac{\partial}{\partial a_3},$$

if

$$\mathbf{a} = \mathbf{i}a_1 + \mathbf{j}a_2 + \mathbf{k}a_3.$$

Then,

$$l\nabla = l_1\nabla_1 + l_2\nabla_2 + l_3\nabla_3;$$

hence, by (48),

$$\begin{aligned} l' = \omega l = \mathbf{i}(l_1\nabla_1 + l_2\nabla_2 + l_3\nabla_3)r_1 \\ + \mathbf{j}(\quad \text{as above} \quad)r_2 \\ + \mathbf{k}(\quad \text{as above} \quad)r_3, \end{aligned}$$

so that the operator in question becomes:

$$\omega = \begin{vmatrix} \nabla_1 r_1 & \nabla_2 r_1 & \nabla_3 r_1 \\ \nabla_1 r_2 & \nabla_2 r_2 & \nabla_3 r_2 \\ \nabla_1 r_3 & \nabla_2 r_3 & \nabla_3 r_3 \end{vmatrix}$$

Instead of this we may write also

$$\omega_{\iota\kappa} = \nabla_\kappa r_\iota; \quad \iota, \kappa = 1, 2, 3. \tag{51a}$$

The coefficients of the conjugate operator ω' are obtained from those of ω by inverting the order of the indices; thus

$$\omega'_{\iota\kappa} = \nabla_\iota r_\kappa; \quad \iota, \kappa = 1, 2, 3.$$

Hence $\Omega = \frac{1}{2}(\omega + \omega')$, by (XIII.), will be determined by the coefficients

$$\Omega_{\iota\iota} = \nabla_\iota r_\iota, \quad \Omega_{\iota\kappa} = \tfrac{1}{2}(\nabla_\kappa r_\iota + \nabla_\iota r_\kappa) = \Omega_{\kappa\iota},$$

and the vector $\mathbf{c} = \mathbf{i}(\omega_{32} - \omega_{23}) + \dots$ will be, by (51a):

$$\mathbf{c} = \mathbf{i}(\nabla_2 r_3 - \nabla_3 r_2) + \mathbf{j}(\nabla_3 r_1 - \nabla_1 r_3) + \mathbf{k}(\nabla_1 r_2 - \nabla_2 r_1);$$

but the sum of the terms on the right side is the *curl* of the vector $\mathbf{r}$ taken in the space $\mathbf{a}$; thus

$$\mathbf{c} = \operatorname{curl}_{(a)} \mathbf{r},$$

where $_{(a)}$ is an abbreviation for 'relatively to the space $\mathbf{a}$,' that is

$$\mathbf{c} = \mathbf{i}\left(\frac{\partial r_3}{\partial a_2} - \frac{\partial r_2}{\partial a_3}\right) + \dots .$$

Relatively to the space $\mathbf{r}$ we should have curl $\mathbf{r} = 0$, *i.e.*

$$\operatorname{curl}_{(r)} \mathbf{r} = 0$$

identically, since all $\mathbf{r}$'s are radial vectors, drawn from the common origin O, so that $\mathbf{r}$ is irrotational; using the coordinates r_1, r_2, r_3 we should have simply $\partial r_3/\partial r_2 = 0$, etc., or $\operatorname{curl}_{(r)} \mathbf{r} = 0$ again.

All this is to justify the index $_{(a)}$, in the above expression of $\mathbf{c}$. It must be remembered that our ∇ has also been an abbreviation for $\nabla_{(a)}$, and the *curl* is nothing else than $V\nabla$. Hence, both are to be taken in regard to the space $\mathbf{a}$. But now, after these explanations, we shall omit the index $_{(a)}$, and assume it tacitly without writing it, both for *curl* and ∇.

The *curl* so understood, applied to $\mathbf{a}$ itself, gives, of course, curl $\mathbf{a} = 0$ identically; hence, if $\mathbf{D} = \mathbf{r} - \mathbf{a}$ is the displacement, as in (49), we shall have

$$\mathbf{c} = \operatorname{curl} \mathbf{r} = \operatorname{curl} \mathbf{D}. \tag{51}$$

Having determined both the symmetrical operator Ω and the vector $\mathbf{c}$, we can now apply fully the theorem (XIII.).

Thus, we have ultimately as the expression of the relation between the deformed line-element $\mathbf{l}'$ *and the original line-element* $\mathbf{l}$:

$$\mathbf{l}' = \omega \mathbf{l} = \Omega \mathbf{l} + \tfrac{1}{2} V \mathbf{c} \mathbf{l}, \tag{50}$$

where the vector $\mathbf{c}$ *is given by*

$$\mathbf{c} = \mathit{curl}\ \mathbf{D} \tag{51}$$

and the symmetrical linear operator Ω *by its six coefficients*

$$\Omega_{\iota\iota} = \nabla_\iota r_\iota ; \quad \Omega_{\iota\kappa} = \tfrac{1}{2}(\nabla_\iota r_\kappa + \nabla_\kappa r_\iota) = \Omega_{\kappa\iota}, \tag{52}$$

for $\iota, \kappa = 1, 2, 3$. *Each differential operator is to be taken with respect to the space* $\mathbf{a}$. *The vectors* $\mathbf{D}$ *and* $\mathbf{r}$ *are connected by the simple relation*

$$\mathbf{D} = \mathbf{r} - \mathbf{a}. \tag{49 bis}$$

The interpretation of this result for the case of an *infinitesimal* c is seen at a glance.

The term $\frac{1}{2} V \mathbf{c} \mathbf{l}$ expresses then a **rotation** of the line-element $\mathbf{l}$ by an angle $\frac{1}{2}c$ about the axis $\mathbf{c}$; it is this rotation which is given, in amount and direction, according to (51), by the vector

$$\tfrac{1}{2} \operatorname{curl} \mathbf{D},$$

$\mathbf{D}$ being the displacement of a point. The angle and the direction of the axis of rotation will, of course, be generally different for different $\mathbf{a}$'s, *i.e.* for different parts of the strained body.

If $$\operatorname{curl} \mathbf{D} = 0,$$

the strain is called **irrotational**. This does not mean that no line-element $\mathbf{l}$ experiences a change of direction, but only that no *volume*-element, as a whole, does experience a rotation,—as will be better understood from the following explanation.

The symmetrical part of the operator ω in eq. (50), *i.e.* the operator Ω, produces by itself a change, not only in the length, but generally also in the *direction* of a line-element $\mathbf{l}$. Hence, if also curl $\mathbf{D} = 0$, then *some particular* line-elements only *do not* experience a change of direction. Ω, being symmetrical, has in general *three principal axes*, mutually perpendicular, and three corresponding *principal values* Ω_1, Ω_2, Ω_3. Only those line-elements which before the deformation coincided with one of these axes will conserve their direction notwithstanding the deformation. If the values Ω_1, Ω_2, Ω_3 are all different from one another, then only three such special directions will exist, but at any rate *not less than three*, mutually perpendicular, directions. Let us imagine a bundle of (∞^2) line-elements diverging from a given point $\mathbf{a}$ in all directions, but having, before the deformation, one and the same length l, so that the surface $l = \text{const.}$ will be the boundary of a volume-element of the body, initially spherical. Now, by the deformation, this sphere will become an *ellipsoid*, of which the principal axes will coincide with the principal axes of the operator Ω. Hence, if curl $\mathbf{D} = 0$, the line-elements which coincide with the axes of this ellipsoid initially will remain coincident with them after the deformation. Thus, we see that in this case the elementary sphere will be transformed into an ellipsoid, that generally the diameters of the sphere will experience a change of direction, *with the exception however of three mutually perpendicular diameters*, which will maintain their direction and, consequently, that 'the sphere as a whole will not be turned.'

This is the reason why a strain satisfying the condition

$$\mathbf{c} = \operatorname{curl} \mathbf{D} = 0$$

is called **irrotational**.

Returning now to the more general case, *i.e.* for $c \neq 0$, we may account fully for any strain by saying that *an elementary sphere of radius l*

1° *is transformed into an ellipsoid, of which the semiaxes are*

$$\Omega_1 l, \quad \Omega_2 l, \quad \Omega_3 l.$$

2° *experiences the rotation* $\frac{1}{2}curl\,\mathbf{D}$,*

3° *experiences the translational displacement* **D**.

All that is here enumerated (*i.e.* the shape, the relative dimensions, and the orientation of the axes of the ellipsoid, the direction and amount of rotation and, finally, the displacement itself) will generally be different in different regions of the strained body, *i.e.* for different **a**'s.

Only in special cases will the directions of the principal axes and the corresponding principal values Ω_1, etc., be equal for the entire strained body, or independent of **a**. Then the strain is called **homogeneous**. But generally it will be *heterogeneous*.

Let us return for a moment to the formula (52) for Ω, in order to substitute therein the displacement $\mathbf{D} = \mathbf{r} - \mathbf{a}$ instead of **r**. We have $r_1 = a_1 + D_1$, etc., whence

$$\nabla_1 r_1 = \nabla_1 a_1 + \nabla_1 D_1 = 1 + \nabla_1 D_1,$$

with similar expressions for $\nabla_2 r_2$, $\nabla_3 r_3$; again, for *different* indices, $\nabla_1 a_2 = \nabla_2 a_1 = 0$, etc.; hence

$$\left.\begin{aligned} &\Omega_{\iota\iota} = 1 + \nabla_\iota D_\iota, \\ &\Omega_{\iota\kappa} = \tfrac{1}{2}(\nabla_\iota D_\kappa + \nabla_\kappa D_\iota) = \Omega_{\kappa\iota}, \\ &\iota, \kappa = 1, 2, 3. \end{aligned}\right\} \qquad (52a)$$

These formulae express precisely the same thing as (52).

Written at full length, they are:

$$\Omega_{11} = 1 + \frac{\partial D_1}{\partial a_1}; \quad \Omega_{23} = \frac{1}{2}\left(\frac{\partial D_3}{\partial a_2} + \frac{\partial D_2}{\partial a_3}\right) = \Omega_{32};$$

$$\Omega_{22} = 1 + \frac{\partial D_2}{\partial a_2}; \quad \Omega_{31} = \frac{1}{2}\left(\frac{\partial D_1}{\partial a_3} + \frac{\partial D_3}{\partial a_1}\right) = \Omega_{13};$$

$$\Omega_{33} = 1 + \frac{\partial D_3}{\partial a_3}; \quad \Omega_{12} = \frac{1}{2}\left(\frac{\partial D_2}{\partial a_1} + \frac{\partial D_1}{\partial a_2}\right) = \Omega_{21};$$

a_1, a_2, a_3 being the rectangular coordinates of a point in 'the space **a**,' *i.e.* of a particle of the unstrained body.

* If it be supposed, for the sake of this interpretation, that c, *i.e.* the tensor of curl **D**, is *infinitesimal*. As to Ω_1, Ω_2, Ω_3, they may have any *finite* values.

Let us now consider a *homogeneous* strain coinciding with the given heterogeneous strain in the elementary region round the point **a** or (as mathematicians say) a homogeneous strain '*tangent*' to the given strain in this point. Then the *principal axes* of the operator Ω will have everywhere *the same directions*, so that along them may be taken the normal unit vectors **i**, **j**, **k** and they may be considered as axes of reference for the whole body. Then

$$\left.\begin{aligned} &\Omega_{\iota\iota} = 1 + \nabla_\iota D_\iota; \quad \iota = 1, 2, 3; \\ &\Omega_{\iota\kappa} = 0; \quad \iota \neq \kappa = 1, 2, 3. \end{aligned}\right\} \qquad (53)$$

Moreover, the principal values $\Omega_1 = \Omega_{11} = 1 + \nabla_1 D_1$, etc., will be constant, the strain being supposed homogeneous.

In the case of a *homogeneous* strain, $\partial D_1/\partial a_1 = \Omega_1 - 1$ being constant throughout the body, we shall have (omitting an arbitrary additive function of a_2, a_3 alone),

$$D_1 = (\Omega_1 - 1)a_1,$$

and similarly

$$D_2 = (\Omega_2 - 1)a_2,$$

$$D_3 = (\Omega_3 - 1)a_3,$$

that is simply

$$\mathbf{D} = \Omega\mathbf{a} - \mathbf{a},$$

and, remembering that $\mathbf{r} = \mathbf{D} + \mathbf{a}$,

$$\mathbf{r} = \Omega\mathbf{a}.$$

But Ω is a distributive operator; hence, if $\mathbf{L} = \mathbf{a}_2 - \mathbf{a}_1$ be a straight line of any *finite* length, joining the particles 1, 2 in the unstrained

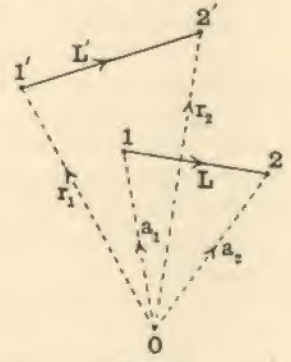

FIG. 34.

body, and if $\mathbf{L}' = \mathbf{r}_2 - \mathbf{r}_1$ be what this line becomes in the strained body (Fig. 34), we shall have

$$\mathbf{L}' = \Omega\mathbf{L}.$$

Observe that, Ω being a linear operator, constant throughout the body, not only do the points $1'$, $2'$ correspond to 1 and 2 respectively, but also the straight line joining $1'$ and $2'$ corresponds to the whole line 12, *i.e.* $\overline{1'2'}$ is composed of the same particles as $\overline{12}$, or also: any straight line *remains straight*. It is *this* property which is particularly characteristic of a *homogeneous* strain.

Now, let us return again to the general case of heterogeneous strain.

To get the elongation of any linear element, we have only to consider the purely irrotational part of the strain; thus, writing

$$\mathbf{l}' = \Omega\mathbf{l},$$

we shall have, by (53) and (xv.):

$$\mathbf{l}' = \mathbf{i}(1+\nabla_1 D_1)l_1 + \mathbf{j}(1+\nabla_2 D_2)l_2 + \mathbf{k}(1+\nabla_3 D_3)l_3. \qquad (54)$$

By **elongation** (*i.e.* relative elongation) of a line-element is meant the ratio

$$\lambda = \frac{l'-l}{l} = \frac{l'}{l} - 1.$$

Hence, **the principal elongations** will be

$$\lambda_1 = \nabla_1 D_1, \quad \lambda_2 = \nabla_2 D_2, \quad \lambda_3 = \nabla_3 D_3, \qquad (55)$$

where D_1, etc., are the components of displacement taken along the principal axes of Ω or of the strain characterised by Ω.

Using ordinary symbols, (55) would be:

$$\lambda_1 = \frac{\partial D_1}{\partial a_1}, \text{ etc.}$$

If the strain be heterogeneous, it must be remembered that the lines along which a_1, a_2, a_3 are measured are not straight, but generally curved lines which may be considered as the limits of chains consisting of the corresponding principal axes;* we have then a threefold curvilinear, orthogonal system, and the indices 1, 2, 3 are symbols of components taken tangentially to such lines, passing through the given point **a**.

Similarly, if $\mathbf{c} = \operatorname{curl}\mathbf{D} \neq 0$, the *rotational lines*, *i.e.* the lines showing everywhere the direction of the axis of rotation, will generally be curved lines. These lines may traverse the triple system of *elongational lines*, generally, in all imaginable directions.

*The homogeneous strains 'tangent' to the given heterogeneous strain being different for different points **a**.

Having learned the meanings of the different terms and having become familiar with the operator Ω and its companion $V\mathbf{c}$, we shall henceforth make free use of the short formula (50).

Let us imagine three non-coplanar line-elements **l**, **m**, **n** diverging from one and the same point **a** (Fig. 35); these define an elementary

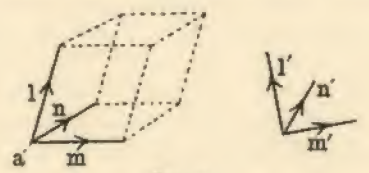

FIG. 35.

parallelopiped, of which the volume $d\tau$ before the deformation is given by

$$d\tau = \mathbf{l}V\mathbf{mn}. \qquad (56)$$

After the deformation of the body, let **l**, **m**, **n** become **l'**, **m'**, **n'**, respectively, and the volume of the parallelopiped $d\tau'$; then

$$d\tau' = \mathbf{l}'V\mathbf{m}'\mathbf{n}'.$$

It is required to compare $d\tau'$ with the original volume $d\tau$.

Let again the tensor of **c** be infinitesimal; then the part $\frac{1}{2}V\mathbf{c}$ of the operator ω in (50), representing a pure rotation of every line-element (with the point **a** as origin) by one and the same angle $\frac{1}{2}c$ about one and the same axis, will, of course, exercise no influence on the volume of the parallelopiped. In other words, the change of volume will be the same as if c were equal to zero. Thus, it will suffice to take into account the symmetrical part of the operator only, and to write

$$\mathbf{l}' = \Omega\mathbf{l}, \quad \mathbf{m}' = \Omega\mathbf{m}, \quad \mathbf{n}' = \Omega\mathbf{n};$$

hence

$$\frac{d\tau'}{d\tau} = \frac{\Omega\mathbf{l}V\Omega\mathbf{m}\Omega\mathbf{n}}{\mathbf{l}V\mathbf{mn}}. \qquad (57)$$

Now, this ratio of scalarly-vectorial products is an *invariant* characterising Ω itself, *i.e.* it does not change at all if instead of **l**, **m**, **n** any other set of three (non-coplanar) elementary vectors, diverging from the same point **a**, be taken. This property belongs also to the *general* linear operator ω, *i.e.*

$$x = \frac{\omega\mathbf{l}V\omega\mathbf{m}\omega\mathbf{n}}{\mathbf{l}V\mathbf{mn}}$$

is an invariant of ω.

This theorem is proved in the following way:*

*Cf. Heaviside, *loc. cit.*

The most general vector may be expressed by $\alpha\mathbf{l}+\beta\mathbf{m}+\gamma\mathbf{n}$, α, β, γ being freely variable scalars. Now, if we take $\alpha\mathbf{l}$ instead of $\mathbf{l}$, both the numerator and the denominator will be simply multiplied by α, so that the quotient x will not be changed; again, adding $\beta\mathbf{m}$, we get in the numerator the complementary term $\beta.\omega\mathbf{m}\mathrm{V}\omega\mathbf{m}\omega\mathbf{n}$, which vanishes identically, by (VII.), and similarly also in the denominator $\beta.\mathbf{m}\mathrm{V}\mathbf{m}\mathbf{n}=0$; hence x will still be unaltered. And it will remain invariable if the term $\gamma\mathbf{n}$ be added. Hence, if $\mathbf{l}$ be changed in the most general way, the value of x will not change; and, similarly, if instead of $\mathbf{m}$, $\mathbf{n}$ any other vectors be taken, x will still keep its value. Q.E.D.

Applying the theorem to Ω (as a special kind of ω), we see by (57) that the ratio $d\tau'/d\tau$ is independent of the dimensions and of the shape of the volume-element $d\tau$, and depends only on the properties of Ω, *i.e.* on the properties of the given strain, at the place where $d\tau$ is taken.

Hence, the ratio

$$\theta=\frac{d\tau'-d\tau}{d\tau}=\frac{d\tau'}{d\tau}-1,$$

which is called the **cubic dilatation**, or simply **the dilatation**, has a perfectly definite meaning, requiring no further explanatory specifications. We can speak simply of 'the dilatation *at a point*.'

Now, the value of the ratio (57) being independent of the choice of the line-elements $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$, these may be, most conveniently, taken along the principal axes of Ω at the given point; hence, denoting the directions of these axes by $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, and taking

$$\mathbf{l}=l\mathbf{i},\quad \mathbf{m}=m\mathbf{j},\quad \mathbf{n}=n\mathbf{k},$$

so that $\Omega\mathbf{l}=l\Omega_1\mathbf{i}$, etc., we have

$$\frac{d\tau'}{d\tau}=\frac{\Omega_1\mathbf{i}\mathrm{V}\Omega_2\mathbf{j}\Omega_3\mathbf{k}}{\mathbf{i}\mathrm{V}\mathbf{j}\mathbf{k}}=\Omega_1\Omega_2\Omega_3,$$

the principal values Ω_1, Ω_2, Ω_3 being ordinary scalars.

Thus, by (52*a*), the *cubic dilatation* will be:

$$\theta=(1+\nabla_1 D_1)(1+\nabla_2 D_2)(1+\nabla_3 D_3)-1, \tag{58}$$

or, by (55),

$$\theta=(1+\lambda_1)(1+\lambda_2)(1+\lambda_3)-1, \tag{58a}$$

where λ_1, λ_2, λ_3 are the principal linear elongations.

Notice that we have seen already that an elementary sphere of radius l, and consequently of volume $\frac{4}{3}\pi l^3$, is transformed into an ellipsoid having $\Omega_1 l$, $\Omega_2 l$, $\Omega_3 l$ as semiaxes, and consequently the volume

$$\tfrac{4}{3}\pi\Omega_1\Omega_2\Omega_3 l^3,$$

so that, in fact, the ratio of volumes is $\Omega_1\Omega_2\Omega_3$, as above.

Infinitesimal Strain.

A strain is said to be *infinitesimal* if the products and squares of the derivatives $\partial/\partial a_1$, $\partial/\partial a_2$, $\partial/\partial a_3$ of any components of the displacement $\mathbf{D}$ be negligible in comparison with their first powers (after the final rejection of any arbitrary displacement of the *whole body* in space).

Under such conditions, the cubic dilatation becomes, by (58*a*), neglecting $\lambda_1\lambda_2=\dfrac{\partial D_1}{\partial a_1}\dfrac{\partial D_2}{\partial a_2}$ and *a fortiori* $\lambda_1\lambda_2\lambda_3$,

$$\theta=\lambda_1+\lambda_2+\lambda_3, \tag{59a}$$

or, by (58):

$$\theta=\nabla_1 D_1+\nabla_2 D_2+\nabla_3 D_3=\nabla\mathbf{D},$$

i.e. (see Chapter I.):

$$\theta=\mathrm{div}\,\mathbf{D}; \tag{59}$$

thus, *the cubic dilatation is equal to the divergence of the displacement.*

This result may also be obtained more directly, without (58), by simply recurring to the definition of divergence by means of the surface-integral, *i.e.* by the original definition of *div* as given in Chapter I.

For, let τ be the volume of any portion of the body, before its

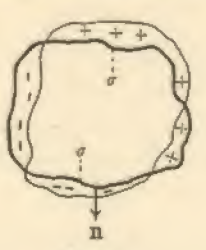

FIG. 36.

deformation, bounded by the surface σ, of which the unit vector $\mathbf{n}$ is the *external* normal, and let τ' be the volume of the same portion

of the body, after the deformation. Then, remembering that $\mathbf{D}$ is supposed to be infinitesimal, we shall have (see Fig. 36),

$$\tau' - \tau = \int \mathbf{Dn} \,.\, d\sigma,$$

whence

$$\theta = \operatorname*{Lim}_{\tau \to 0} \left\{ \frac{1}{\tau} \int \mathbf{Dn} \,.\, d\sigma \right\},$$

i.e. by the cited definition of 'div,'

$$\theta = \operatorname{div} \mathbf{D}.$$

Thus, operating with one and the same *Hamiltonian* ∇ upon the displacement $\mathbf{D}$, vectorially or scalarly, we get the two most characteristic properties of (infinitesimal) deformation, viz.

the rotation of an element $\frac{1}{2}\mathbf{c} = \frac{1}{2} \operatorname{curl} \mathbf{D} = \frac{1}{2} \mathrm{V} \nabla \mathbf{D}$,

the cubic dilatation $\theta = \operatorname{div} \mathbf{D} = \nabla \mathbf{D}$.

For an infinitesimal deformation, all the coefficients of the symmetrical operator

$$\Omega = \begin{vmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} \\ & \Omega_{22} & \Omega_{23} \\ & & \Omega_{33} \end{vmatrix}$$

have a very simple meaning (with an arbitrary system of reference $\mathbf{i}, \mathbf{j}, \mathbf{k}$). We have, in fact, by (52*a*), for equal indices,

$$\Omega_{11} = 1 + \nabla_1 D_1 = 1 + \frac{\partial D_1}{\partial a_1}.$$

Infinitesimals of the second and higher orders being neglected, we see from this that Ω_{11} is the linear elongation of an element taken

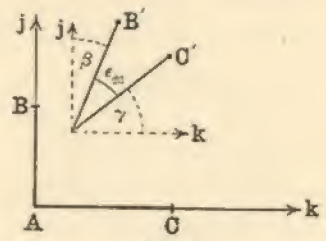

FIG. 37.

initially in the direction $\mathbf{i}$, and that Ω_{22}, Ω_{33} have the same meaning, with $\mathbf{j}$, $\mathbf{k}$ instead of $\mathbf{i}$. Again, for different indices,

$$2\Omega_{23} = \nabla_2 D_3 + \nabla_3 D_2 = \frac{\partial D_3}{\partial a_2} + \frac{\partial D_2}{\partial a_3}, \text{ etc.} \qquad (\mathrm{A})$$

Let two line-elements coincide before the deformation with $\mathbf{j}$, $\mathbf{k}$, as regards direction, thus including a right angle; let the angle included after deformation be ϵ_{23}; then, neglecting magnitudes of the second and higher orders, we see immediately from Fig. 37 that $\nabla_2 D_3 = tg\beta \doteqdot \beta$, and similarly $\nabla_3 D_2 = tg\gamma \doteqdot \gamma$; hence

$$\epsilon_{23} = \frac{\pi}{2} - (\nabla_2 D_3 + \nabla_3 D_2),$$

so that

$$2\Omega_{23} = \frac{\pi}{2} - \epsilon_{23}.$$

Similar meanings will be found for Ω_{31}, Ω_{12} with respect to $\mathbf{k}, \mathbf{i}$; $\mathbf{i}, \mathbf{j}$.

Thus, the expressions in (A) are, for an infinitesimal strain, the **shears** parallel to the planes $\mathbf{j}, \mathbf{k}$; $\mathbf{k}, \mathbf{i}$; $\mathbf{i}, \mathbf{j}$ respectively.

If $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be taken along *the principal axes* of Ω in the given point $\mathbf{a}$, the shears Ω_{23}, etc., vanish, of course, *i.e.* the right angles between *these* axes remain right angles after the deformation.

The Equation of Continuity.

Let ρ be the density of mass of the body, *i.e.* let $\rho\, d\tau$ be the mass of its volume-element $d\tau$, before the deformation. Let us denote by δ the change, due to an *infinitesimal strain*,* of any magnitude belonging to an *individual* element of the body.

Then, the so-called **equation of continuity**, which is the expression of the invariability of the mass of every individual element, will be

$$\delta(\rho\, d\tau) = 0$$

or

$$d\tau \,.\, \delta\rho + \rho\, \delta(d\tau) = 0,$$

or, substituting the dilatation $\theta = \delta(d\tau)/d\tau$,

$$\delta\rho + \rho\theta = 0.$$

Hence, by (59), the equation of continuity may be written

$$\delta\rho + \rho \operatorname{div} \mathbf{D} = 0. \qquad (60)$$

Introducing, as in the beginning, the vector $\mathbf{r}$, and remembering that $\mathbf{D}$ is to be infinitesimal and that $\delta\mathbf{a} = 0$ (because $\mathbf{a}$ itself individualises a given element), we may write $\mathbf{D} = \delta\mathbf{r}$, and consequently, instead of (60),

$$\delta\rho + \rho \operatorname{div} (\delta\mathbf{r}) = 0. \qquad (60a)$$

If $\delta\mathbf{r}$ is to be any *virtual displacement*, then it must in the first place satisfy this condition.

* Henceforth only *infinitesimal* strains will be considered.

Classification of Strains.

If the displacement **D** satisfies throughout the strained body, or throughout the whole 'region **a**' considered, the condition

$$\operatorname{curl} \mathbf{D} = 0,$$

then, as we have seen already, no element $d\tau$ experiences a rotation. Consequently, such a strain has been called **irrotational**, or **longitudinal**. The meaning of the last name will be explained in the next section.

On the other hand, if **D** satisfies, throughout the whole region **a**, the condition

$$\operatorname{div} \mathbf{D} = 0,$$

no element $d\tau$ experiences a change of volume. Such a strain, which will be met with in the study of incompressible elastic solids or liquids, is called a purely **transversal** strain. It might also be called a **solenoidal** or **circuital** strain, these epithets being usually attached to any vector **w**, or better, vector-field **w**, satisfying the condition $\operatorname{div} \mathbf{w} = 0$; but we prefer the name 'transversal strain,' the meaning of which will be cleared up further on.

It is not difficult to prove that every vector-field **w** may be decomposed, and this, moreover, in one *single* way, into a purely irrotational and a purely solenoidal part. Thus, *the most general strain may be represented as a superposition of a longitudinal and a transversal strain.*

In the superposition of any two (or more) infinitesimal strains given, say, by $\mathbf{D}_1$, $\mathbf{D}_2$, *i.e.* in the strain given by $\mathbf{D} = \mathbf{D}_1 + \mathbf{D}_2$, we have a simple *superposition* both of the dilatations and the rotations of any element. For, both *curl* and *div* are distributive operators, so that

$$\theta = \operatorname{div}(\mathbf{D}_1 + \mathbf{D}_2) = \operatorname{div} \mathbf{D}_1 + \operatorname{div} \mathbf{D}_2 = \theta_1 + \theta_2,$$

and similarly

$$\mathbf{c} = \operatorname{curl}(\mathbf{D}_1 + \mathbf{D}_2) = \operatorname{curl} \mathbf{D}_1 + \operatorname{curl} \mathbf{D}_2 = \mathbf{c}_1 + \mathbf{c}_2.$$

If the first strain is purely longitudinal, and the second purely transversal, then $\theta = \theta_1$, $\mathbf{c} = \mathbf{c}_2$, *i.e.* the resultant strain owes its dilatational properties wholly to the first, and its rotational properties entirely to the second strain.

Thus, it is convenient, and possible, to treat separately the two kinds of strain, longitudinal and transversal. Such a splitting is not artificial at all. If the time-development of the phenomena is

considered, then the two kinds of strains are, generally, actually separated, *i.e.* the respective disturbances are propagated with *different velocities*, so that if also emitted by the source simultaneously, the first kind will soon lag behind the second, or *vice versa*.

Longitudinal Strain.

The condition

$$\operatorname{curl} \mathbf{D} = 0$$

being fulfilled everywhere, the displacement has a scalar potential, say ψ, *i.e.*:

$$\mathbf{D} = \nabla\psi. \tag{61}$$

This potential is not necessarily a single-valued function of the position of a point in the region **a**, say, for example, of the coordinates a_1, a_2, a_3. It will be such a function for an *acyclic* or simply-connected region, as for instance for a medium occupying all the (Euclidean) space, or for a sphere, for a cylinder and for similar bodies. But generally it may be a *many-valued* function, viz. for a *cyclic* body, as for an anchor ring.

The displacement is in the present case, by (61), *normal* to the surfaces

$$\psi = \text{const.};$$

for, by (x.), we may write

$$\mathbf{D} = \mathbf{n} \frac{\partial \psi}{\partial n}.$$

Two equipotential surfaces, to which correspond the values ψ and $\psi + d\psi$ of the potential, cut out from the body an infinitely thin sheet or lamina; thus, the whole body may be split into a series of such sheets.

If ϵ is the (infinitesimal) thickness of one of such sheets, in a given place, before the deformation, and ϵ' after; then, putting $\epsilon' - \epsilon = \delta\epsilon$, we shall have

$$\frac{\delta\epsilon}{\epsilon} = \frac{\partial^2 \psi}{\partial n^2}$$

as the formula for the relative thickening of a sheet.

The *cubic dilatation* will be, by (59),

$$\theta = \operatorname{div} \nabla\psi = \nabla\nabla\psi$$

or

$$\theta = \nabla^2 \psi; \tag{62}$$

in ordinary rectangular coordinates, for example, we should have

$$\theta = \frac{\partial^2 \psi}{\partial a_1^2} + \frac{\partial^2 \psi}{\partial a_2^2} + \frac{\partial^2 \psi}{\partial a_3^2}$$

But, in general considerations, it is best to retain the short formula (62) without specifying the choice of any system of coordinates. ∇^2 is, at any rate, quite independent of any system of reference. (See Chap. I.)

Writing again, as in (50), for an arbitrary line-element,

$$\mathfrak{l}' = \omega \mathfrak{l},$$

the *whole* operator ω will, in the present case of irrotational strain, be a *symmetrical* linear operator:

$$\omega = \Omega,$$

for $\mathbf{c} = \operatorname{curl} \mathbf{D} = 0$.

By (52*a*), the coefficients of Ω will now be, with *any* directions of the axes of rectangular coordinates a_1, a_2, a_3,

$$\Omega_{11} = 1 + \frac{\partial^2 \psi}{\partial a_1^2}, \text{ etc.},$$

$$\Omega_{23} = \frac{1}{2}\left(\frac{\partial^2 \psi}{\partial a_2 \partial a_3} + \frac{\partial^2 \psi}{\partial a_3 \partial a_2}\right) = \frac{\partial^2 \psi}{\partial a_2 \partial a_3}, \text{ etc.},$$

so that the *shears* will be given by

$$2\,\frac{\partial^2 \psi}{\partial a_2 \partial a_3}, \text{ etc.}$$

If all the shears vanish for *any* coordinate planes, the strain consists in a *pure change of volume*; for then we have, for any choice of the rectangular coordinate axes,

$$\Omega = \begin{vmatrix} \Omega_{11} & 0 & 0 \\ 0 & \Omega_{22} & 0 \\ 0 & 0 & \Omega_{33} \end{vmatrix}$$

In other words: *any* direction whatever plays in this case the part of a principal axis; hence

$$\Omega_{11} = \Omega_{22} = \Omega_{33}, \text{ say, } = N,$$

so that

$$\mathfrak{l}' = N\mathfrak{l},$$

or $l' : l = N$, independently of the direction of the line-element. Thus, in this particular case, the operator Ω degenerates into an ordinary scalar multiplier.

The reader may verify this result by returning to the equations $\partial^2\psi/\partial a_2 \partial a_3 = 0$, etc., and deducing from them the conclusions which easily suggest themselves.

If the displacement is the same (as regards both absolute value and direction) for all points situated on any plane $a_1 = \text{const.}$ and varies only on passing from one such plane to another parallel plane, or—in other words—if $\mathbf{D}$ depends only on a_1, then

$$\theta = \frac{\partial D_1}{\partial a_1} = \frac{\partial^2 \psi}{\partial a_1^2}.$$

At the same time it follows from the condition $\operatorname{curl} \mathbf{D} = 0$, that

$$\frac{\partial D_3}{\partial a_1} = 0, \quad \frac{\partial D_2}{\partial a_1} = 0,$$

i.e. $D_2 = const.$, $D_3 = const.$ Thus, the transverse components of displacement, *i.e.* the components normal to the axis of a_1, are the same for the whole body, giving by themselves only a displacement of the whole body in space, as if it were a rigid body. But as we are here concerned rather with relative displacements of some parts of the body in regard to others, we may, without impairing the generality, write simply

$$D_2 = D_3 = 0.$$

Thus, what remains will be D_1 only, *i.e.* a purely *longitudinal* displacement. Hence the name given above to strains satisfying the condition $\operatorname{curl} \mathbf{D} = 0$.

Transversal Strain.

If, in the whole body,

$$\theta = \operatorname{div} \mathbf{D} = 0,$$

so that the volume of any of its portions remains invariable, we may put

$$\mathbf{D} = \operatorname{curl} \mathbf{A}, \tag{63}$$

with the supplementary condition for the auxiliary vector $\mathbf{A}$:

$$\operatorname{div} \mathbf{A} = 0. \tag{64}$$

Generally, if a (solenoidal) vector $\mathbf{X}$ is put into the form $\operatorname{curl} \mathbf{Y}$, then $\mathbf{Y}$ is called the **vector-potential** of $\mathbf{X}$. Thus, $\mathbf{A}$ will be the *vector-potential of the displacement* $\mathbf{D}$.

For a longitudinal strain, we had

$$\mathbf{D} = \nabla\psi,$$

whereas now we have, somewhat analogously, writing $\operatorname{curl} = V\nabla$,

$$\mathbf{D} = \nabla V \mathbf{A}.$$

The scalar potential of displacement does not exist for a transversal strain; it is replaced here, *mutatis mutandis*, by the vector-potential.

By (63) we have for the vector $\mathbf{c} = \operatorname{curl} \mathbf{D}$, *i.e.* for twice the rotation of an element $d\tau$,

$$\mathbf{c} = \operatorname{curl}^2 \mathbf{A}.$$

Now, for any vector, say $\mathbf{w}$, we had, in Chapter I., the identical formula

$$\operatorname{curl}^2 \mathbf{w} = \nabla \operatorname{div} \mathbf{w} - \nabla^2 \mathbf{w}. \qquad \text{(XVII.)}$$

In our case, $\operatorname{div} \mathbf{A}$ being $= 0$, we have, more simply,

$$\mathbf{c} = -\nabla^2 \mathbf{A}, \qquad (65)$$

and this is only a vectorial generalisation of the well-known equation of Laplace-Poisson. Using, for example, rectangular components 1, 2, 3, we may split (65) into

$$\nabla^2 A_1 = -c_1, \quad \nabla^2 A_2 = -c_2, \quad \nabla^2 A_3 = -c_3.$$

Now, as is well known from the elementary theory of potential functions,

$$A_1 = \frac{1}{4\pi} \int \frac{c_1}{r} d\tau,$$

with similar expressions for A_2, A_3 (if A_1, etc., and their first derivatives are finite, continuous and vanish 'at infinity' in the well-known way). Hence, recombining again,

$$\mathbf{A} = \frac{1}{4\pi} \int \frac{\mathbf{c}}{r} d\tau, \qquad (66)$$

where $d\tau$ is any volume-element of the body (in which the rotation $\mathbf{c}$ does not vanish) and r the distance from $d\tau$ to that point P, for which $\mathbf{A}$ has to be calculated (see Fig. 38), the integration,

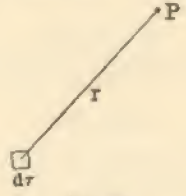

Fig. 38.

of course, being extended to all rotating particles, or, what turns out to be the same thing, to the whole strained body.

To this formula and to the analogous one for the scalar potential,

$$\psi = -\frac{1}{4\pi} \int \frac{\theta}{r} d\tau,$$

which follows from the differential equation (62) precisely in the same way as (66) from (65), we shall return later, namely in the kinematical part of the subject, where not the displacement itself but its velocity or time-rate of change will be treated. We shall then give also the interpretation of these formulae.

At any rate, it must be observed that both the scalar and the vector-potentials are auxiliary notions, while the displacement, the rotation or dilatation have immediate physical meanings.

Supposing again (as for the longitudinal strain) that $\mathbf{D}$, in a transversal strain, depends only on a_1, we have from $\operatorname{div} \mathbf{D} = 0$:

$$\nabla_1 D_1 = \frac{\partial D_1}{\partial a_1} = 0$$

or $D_1 = const.$, so that, by the above remarks, we may put, in this case:

$$D_1 = 0;$$

thus, the remaining displacement is entirely *normal* to the axis of a_1, or

$$\mathbf{Di} = 0.$$

This is the reason why a strain satisfying the condition $\operatorname{div} \mathbf{D} = 0$ has been called a *transversal* strain.

Surfaces of Discontinuity.

Till now the displacement $\mathbf{D}$ has been always assumed to be a continuous function of $\mathbf{a}(a_1, a_2, a_3)$; in other words, it has been supposed that on passing from one particle to others the vector $\mathbf{D}$ changes in a continuous manner.

But now let us admit that at some surfaces (*i.e.* on traversing some surfaces from one side to the other) the displacement itself or its derivatives, or the derivatives of some of its components, may *jump* or experience a discontinuity.

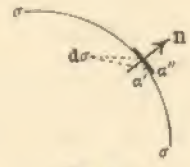

Fig. 39.

Let σ (Fig. 39) represent such a **surface of discontinuity**. Let the unit vector $\mathbf{n}$ be the normal of one of its elements $d\sigma$. If the

equation of this surface, before the deformation of the body, be $f(a_1, a_2, a_3) = 0$, or more briefly,

$$f = f(a) = 0,^*$$

then we may express the normal by

$$\mathbf{n} = \nabla f : \frac{\partial f}{\partial n},$$

since $\frac{\partial f}{\partial n}$ is simply the tensor of ∇f, and ∇f is a vector normal to the surface σ of $f = 0$. (Here the normal $\mathbf{n}$ is meant to be taken in the direction of *increasing* f.)

The jump of any magnitude, say α, whether a scalar or a vector, on traversing σ in the positive direction of $\mathbf{n}$, will be denoted (according to the notation of Christoffel) by $[\alpha]$; thus, using the accents $'$ and $''$ in the manner explained in Fig. 39, we shall write

$$\alpha'' - \alpha' = [\alpha].$$

Consequently, the jump of the displacement will be denoted by $[\mathbf{D}]$.

It is scarcely necessary to remember that a jump of a scalar is a scalar, and the jump of a vector, being simply the difference of two vector values, is a vector.

We shall suppose that the body, occupying a certain portion of space continuously in its 'natural' state, continues to do so in the strained state, viz. that it is not torn by the strain. From this supposition it follows immediately that the *normal* component of the displacement is *continuous*, or that

$$[\mathbf{Dn}] = 0. \qquad (67)$$

On the other hand, the part of the displacement *tangential* to the surface σ may experience a jump; denoting this part, in absolute value and direction, by the vector $\mathbf{T}$, *i.e.* writing

$$\mathbf{T} = \mathbf{D} - \mathbf{n}(\mathbf{Dn}) = \mathrm{V}\mathbf{n}\mathrm{V}\mathbf{Dn},$$

we shall have, generally,

$$[\mathbf{T}] \neq 0.$$

This means, obviously, that two parts of the strained body, without separating from, may glide along one another, and this gliding is expressed, at any point of the surface of contact, by $[\mathbf{T}]$, which—being a vector—may be denoted by $\mathbf{S}$.

* Here 'a' is not the tensor of $\mathbf{a}$ but an abbreviation for 'a_1, a_2, a_3' or for any other three parameters individualising a material particle.

The normal component of $\mathbf{D}$ being continuous, we may say also that the gliding or the jump of the tangential part represents the entire jump of the displacement:

$$[\mathbf{D}] = [\mathbf{T}] = \mathbf{S}.$$

Remembering the definition of *curl* in terms of the line-integral, the reader will easily prove that a surface of discontinuity of the tangential part of the vector in question is equivalent to an indefinitely thin 'rotational' sheet, *i.e.* to a sheet of curl $\mathbf{D} = 2\mathbf{c}$ with an indefinitely large value of c, but so that the product of the thickness h by c, or the moment or strength of the rotational sheet, is finite,—the direction of curl $\mathbf{D}$ being normal to the plane $\mathbf{n}$, $\mathbf{S}$, *i.e.* tangent to the surface σ and normal to the gliding. The exact expression of this equivalence is

$$\mathrm{Lim}\,(h\,\mathrm{curl}\,\mathbf{D}) = \mathrm{V}\mathbf{n}\mathbf{S} = \mathrm{V}\mathbf{n}[\mathbf{T}],$$

where h, the thickness of the sheet, tends to zero.

$\frac{1}{2}S$ is the *strength* of the rotational sheet, per unit length, measured in the direction of $\mathbf{S}$.

The interpretation of this equivalence is obvious. In fact, instead of the gliding of two bodies in mutual contact may be substituted (in its purely kinematical aspect, of course) their rolling on very thin tubes of circular section, interposed between the bodies and *rotating* about their own axes, as shown in Fig. 40, the plane of this

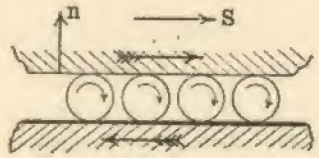

Fig. 40.

figure being $\mathbf{n}$, $\mathbf{S}$. The axes of these tubes, not necessarily straight, will coincide with the rotational lines, *i.e.* with the lines indicating everywhere the direction of the vector $\mathbf{c} = \frac{1}{2}\,\mathrm{curl}\,\mathbf{D}$.

In regard to this equivalence of surfaces of gliding and of sheets of rotation, the following remark may not be superfluous.

When we say that a given strain is purely irrotational (or 'longitudinal') in the whole body or medium in question, we mean that not only curl $\mathbf{D} = 0$ in continuous regions but also that there are no infinitesimally thin sheets of rotation or surfaces of gliding, *i.e.* that not only curl $\mathbf{D} = 0$, where continuity reigns, but also $[\mathbf{T}] = 0$ and

consequently $[\mathbf{D}]=0$ at any surfaces of discontinuity. The condition that the body should not be torn requires the continuity of the normal component, and the supposed complete irrotationality of the strain requires the continuity of the tangential part of the displacement; thus, the whole displacement must be continuous, $[\mathbf{D}]=0$, for irrotational strains. Nevertheless, certain surfaces of discontinuity are possible for such strains; in fact, the displacement $\mathbf{D}$ itself must then be continuous, but different magnitudes deduced from it, say, by the process of differentiation in regard to space, may be discontinuous.

Such discontinuities have played, since the times of Riemann, a very considerable part in many branches of mathematical physics.

In order to become acquainted in the easiest way possible with some of their fundamental properties, let us begin with the simpler case of a scalar. The passage to discontinuities of a vector will then follow almost immediately.

Let ϕ be any *scalar* function of position in the space $\mathfrak{a}$; suppose that, through the surface σ, the function ϕ remains continuous, *i.e.*

$$[\phi]=0,$$

but that its *first* derivatives in regard to space, say $\partial\phi/\partial a_1$, etc., and consequently also its slope $\nabla\phi$, is discontinuous, or that

$$[\nabla\phi]\neq 0.$$

Then the discontinuity is called a *discontinuity of the first order*. Similarly, if $\left[\frac{\partial\phi}{\partial t}\right]\neq 0$. (The time, however, will come in only in the next section of this chapter.) Generally, if the function ϕ itself and all its derivatives up to the $(n-1)^{st}$ order, inclusively, are continuous, and the derivatives of the nth (with or without the derivatives of higher) order become discontinuous, then the discontinuity is called *of the nth order*.

The mere fact that the discontinuities are supposed to be *distributed over some surface* σ, or $f=0$, leads of itself to certain conditions, which Hadamard calls the **identical conditions.***

Limiting ourselves to the consideration of discontinuities of the *first order* only, we shall now proceed to develop the corresponding identical conditions.

*J. Hadamard, *Leçons sur la propagation des ondes et les équations de l'hydrodynamique*, Paris, 1903, p. 81. Compare also G. Zémplen, *Encyklop. d. math. Wissenschaften*, Vol. IV.$_2$ Heft. 3, 1906, and P. Appell, *Traité de mécanique*, Vol. III. Chap. XXXIII. iii. 1903.

Let, then,
$$[\phi]=0, \tag{68}$$
but
$$[\nabla\phi]\neq 0, \tag{69}$$
at the surface σ. This surface divides the space into two regions; we shall suppose that in each of these not only the gradient $\nabla\phi$ exists and is continuous, but also that on approaching the surface σ from one or from the other side, $\nabla\phi$ and consequently also $\partial\phi/\partial a_1$, etc., tend to definite values, which however are different for the two sides, viz. $(\nabla\phi)'$, $(\nabla\phi)''$, respectively.

Under such conditions it may easily be shown that the differentials of both ϕ' and ϕ'' taken along any line-element *lying on the surface* σ are expressed by the ordinary rules of the calculus (see Hadamard, *loc cit.* Art. 72).

Let, then, $\mathfrak{l}$ be such a line-element, having an arbitrary direction *on the surface* of discontinuity. Then, at the one side of the surface,

$$d\phi'=\mathfrak{l}\nabla\phi',$$

and similarly, at the opposite side,

$$d\phi''=\mathfrak{l}\nabla\phi'',$$

whence, by subtraction, and remembering that $[\phi]=0$ on the whole surface, and consequently $d[\phi]=d\phi'-d\phi''=0$,

$$\mathfrak{l}[\nabla\phi]=0.$$

This equation is true for *every* $\mathfrak{l}$ lying on σ; hence:

> *The vector* $[\nabla\phi]$ *or the jump of the vector* $\nabla\phi$ *is normal to the surface of discontinuity* (provided always, as has been supposed, that ϕ itself remains continuous).

Denoting, then, by Λ a scalar quantity, the values of which will generally be different for different points of σ, we may write

$$[\nabla\phi]=\Lambda\mathfrak{n} \tag{70}$$

or, remembering that $\mathfrak{n}=\nabla f:\partial f/\partial n$,

$$[\nabla\phi]=\lambda\nabla f, \tag{70a}$$

λ being again a scalar, viz. $\lambda=\Lambda:\partial f/\partial n$, and $f=0$ the equation of the surface.

The equation (70) or (70*a*) is the vector equivalent of Hadamard's three scalar *identical conditions* for a discontinuity of the first order.

The value of the scalar Λ, *i.e.* the *absolute value of the jump*, remains arbitrary; but if it be given, all about the discontinuity is known, since the *direction* of the jump is determined, being, by the above theorem, everywhere normal to the given surface $f=0$.

Of course, it has been tacitly assumed here that this surface has at the points considered a determinate normal.

The above theorem may now be applied to each of the components of any vector, for instance, of the displacement **D**. If all the components of **D** itself are continuous, *i.e.* if

$$[D_1]=[D_2]=[D_3]=0,$$

and hence also

$$[\mathbf{D}]=0, \tag{71a}$$

then we have, by (70), writing simply $\phi=D_1$, or D_2, or D_3, and denoting by m_1, m_2, m_3 three (mutually independent) scalars, every one of which takes in turn the place of the above A,

$$\left.\begin{aligned}[\nabla D_1]&=m_1\mathbf{n},\\ [\nabla D_2]&=m_2\mathbf{n},\\ [\nabla D_3]&=m_3\mathbf{n},\end{aligned}\right\} \tag{71'}$$

where we may write also $\mathbf{n}=\nabla f:\partial f/\partial n$.

The three vector equations are equivalent to Hadamard's nine scalar identical conditions for a discontinuity of the first order of a vector.

Now, the scalars m_1, m_2, m_3 may be considered as the components of a vector, say $\mathbf{m}$:

$$\mathbf{m}=\mathrm{i}m_1+\mathrm{j}m_2+\mathrm{k}m_3,$$

this vector being characteristic for the given discontinuity. In fact, if we know the form of the surface $f=0$ and the vector $\mathbf{m}$ for every point of this surface, then we know everything about the discontinuity. The reader will easily show that $\mathbf{m}$ is independent of the choice of the system of reference (and it is this that proves its true vectorial character). Consequently we have no need of such a system, and thus instead of the three equations (71') we may write a single equation. In fact, let $\mathbf{x}$ be any vector whatever, say a unit vector, of a quite arbitrary direction; then we may write instead of (71'):

$$[\nabla(\mathbf{Dx})]=\mathbf{n}(\mathbf{mx}). \tag{71}$$

We here reach the extreme of condensation, viz. *nine* scalar identical conditions of Hadamard packed closely into a single formula.

From this the formulae for the jump of the (double) rotation or of $\mathbf{c}=\operatorname{curl}\mathbf{D}$ and for the jump of the cubic dilatation $\theta=\operatorname{div}\mathbf{D}$ may immediately be deduced.

In fact, $\operatorname{div}\mathbf{D}=\nabla_1 D_1+\nabla_2 D_2+\nabla_3 D_3=\nabla\mathbf{D}$ and

$$\operatorname{curl}\mathbf{D}=\mathrm{i}(\nabla_2 D_3-\nabla_3 D_2)+\ldots=\mathrm{V}\nabla\mathbf{D};$$

hence

$$[\theta]=[\operatorname{div}\mathbf{D}]=\mathbf{nm}, \tag{72}$$

$$[\mathbf{c}]=[\operatorname{curl}\mathbf{D}]=\mathrm{V}\mathbf{nm}. \tag{73}$$

These formulae show an instructive analogy to one another, especially if they are written in a slightly different way,

$$[\nabla\mathbf{D}]=\mathbf{nm},$$

$$[\mathrm{V}\nabla\mathbf{D}]=\mathrm{V}\mathbf{nm}.$$

From (72), (73) the truth of the following propositions is seen immediately:

1°. If the vector $\mathbf{m}$ be *tangent* to the surface of discontinuity, then $\mathbf{mn}=0$ and

$$[\theta]=[\operatorname{div}\mathbf{D}]=0,$$

i.e. the cubic dilatation remains continuous.

2°. For *any* direction of $\mathbf{m}$, (73) multiplied scalarly by $\mathbf{n}$ gives:

$$[\mathbf{nc}]=[\mathbf{n}\operatorname{curl}\mathbf{D}]=0;$$

the normal component of the rotation remains continuous, so that only its tangential part may jump.

3°. Similarly, multiplying by $\mathbf{m}$,

$$[\mathbf{mc}]=[\mathbf{m}\operatorname{curl}\mathbf{D}]=0,$$

i.e. the component of the rotation taken along $\mathbf{m}$ *also remains continuous.*

Thus, the whole jump of the rotation, considered as a vector, is *normal to the plane* $\mathbf{m}$, $\mathbf{n}$.

If $\mathbf{m}$ be given, both $[\theta]$ and $[\mathbf{c}]$, or $[\operatorname{div}\mathbf{D}]$ and $[\operatorname{curl}\mathbf{D}]$, are given; conversely, if $[\theta]$ and $[\mathbf{c}]$ are known, also $\mathbf{m}$, the vector characteristic of the discontinuity of 1st order, is completely known, since $[\theta]$ determines its normal and $[\mathbf{c}]$ its tangential part.

Kinematics of a Deformable Body.

Until now the strain has been considered without regard to the interval of time in which it has been produced. The displacement $\mathbf{D}$, which in the last sections has been supposed infinitesimal, was simply the difference of the vector $\mathbf{r}$ determining the position of a given particle 'after deformation' and the vector $\mathbf{a}$ determining the position of the same particle 'before deformation,' independently

of the time-interval that has elapsed. Let us now bring in the time t.

The infinitesimal displacement, till now denoted by $\mathbf{D}$, will henceforth be written $d\mathbf{D}$ and considered as taking place in the infinitesimal time-interval dt. Since $\mathbf{a}$ characterises a given individual particle of the body, so that $d\mathbf{a}=0$ ($\mathbf{a}$ being simply the vector defining the 'initial' position of the particle, say for $t=t_0$), we may write $d\mathbf{r}$ instead of $d\mathbf{D}$, remembering that we had generally $\mathbf{D}=\mathbf{r}-\mathbf{a}$.

The infinitesimal vector

$$d\mathbf{D}=d\mathbf{r}$$

will express the element of the *path* described by the individual particle $\mathbf{a}$ during the interval of time $t \to t+dt$.

Thus, denoting by $\mathbf{v}$ the instantaneous *velocity* of motion of the particle, we shall have

$$\mathbf{v}=\frac{d\mathbf{D}}{dt}=\frac{d\mathbf{r}}{dt}.$$

The following distinction is very important and must be constantly kept in mind.

Let ψ be any (scalar or vector) quantity belonging to a given particle of the body. Then, following the individual history of the particle, viz. watching it in its motion, we shall denote the time-rate of change of ψ by

$$\frac{d\psi}{dt},$$

and call it the **individual change**, per unit time. Thus, for example, the velocity $\mathbf{v}$ of a particle is the individual change of the vector $\mathbf{r}$ (per unit time), and has therefore been denoted by $\frac{d\mathbf{r}}{dt}$.

On the other hand, instead of following a given particle, we may concentrate our attention on a fixed point of space,* through which is passing now this particle and now that, and observe the value of ψ reigning at this point, without troubling ourselves about the question to which particles the ψ belongs. The change thus defined may be called the **local change**; per unit time, it will be denoted by

$$\frac{\partial\psi}{\partial t}.$$

* That is on a point given relatively to some system of reference which does not participate in the motion of the deformable body.

For instance, $\frac{\partial\mathbf{v}}{\partial t}$ will be the local rate of change of velocity, while $\frac{d\mathbf{v}}{dt}$ will be the change (per unit time) of velocity of motion of a given particle or the *acceleration* of this individual particle.

Similarly, $\frac{\partial\rho}{\partial t}$ will be the local rate of change of density and $\frac{d\rho}{dt}$ the rate of change of density of a given volume-element of the body, which consists always of the same particles.

The two rates of change, thus distinguished, are connected by a very simple relation. For, in the time dt a given particle experiences the displacement $\mathbf{v}\,dt$, and thus passes from a place where ψ reigns to another place where the value $\psi+dt(\mathbf{v}\nabla)\psi$ would reign if there were no local changes in time; thus, in the most general case we have:

1°. the change $dt(\mathbf{v}\nabla)\psi$

and

2°. the local change $\frac{\partial\psi}{\partial t}dt$.

Now, the individual change in the infinitesimal time dt is equal to the sum of these two changes;* hence, per unit time,

$$\frac{d\psi}{dt}=\frac{\partial\psi}{\partial t}+(\mathbf{v}\nabla)\psi, \tag{74}$$

which is the required relation.

If ψ be a scalar, the parenthesis at $\mathbf{v}\nabla$ may be omitted as superfluous; but should ψ be a vector, we must keep it, to avoid misunderstanding.

By this general formula we have, for instance, for the acceleration of a particle which we follow in its motion:

$$\frac{d\mathbf{v}}{dt}=\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\nabla)\mathbf{v}. \tag{74a}$$

If the velocity be very small, the second term on the right side may be neglected in the presence of the first (as indeed is done in many hydrodynamical problems). But, in general, this term must be retained, so that the acceleration of a particle will be different from the local rate of change of the velocity.

Similarly we have for the rate of individual change of density

$$\frac{d\rho}{dt}=\frac{\partial\rho}{\partial t}+\mathbf{v}\nabla\rho, \tag{74b}$$

a formula which we shall soon use.

* For, in thus combining the two kinds of change which in reality go on simultaneously, we only neglect infinitesimal terms *of second and higher orders*.

We may now in all our previous formulae substitute in place of the infinitesimal displacement (**D**, which now is denoted by $d\mathbf{D}$) the velocity

$$\mathbf{v} = \frac{d\mathbf{D}}{dt}.$$

Let us begin with the first fundamental formula, *i.e.* (48′). Instead of **D**, we have now to write $d\mathbf{D}$ or $\mathbf{v}\,dt$; hence

$$1' - 1 = (1\nabla)\,\mathbf{v}\,.\,dt,$$

where dt, a simple scalar factor, is not exposed, of course, to the action of the operator 1∇.

Now, remembering that 1 is the original line-element and 1′ *the same* element after deformation, and that this deformation has taken place during the time dt, we see that $1' - 1$ is the individual change of 1 or, in the accepted notation, $1' - 1 = d1$. Hence, per unit time,

$$\frac{d1}{dt} = (1\nabla)\,\mathbf{v}. \tag{75}$$

It must be kept in mind that 1, on both sides of (75), is an infinitesimal vector, whereas the velocity $\mathbf{v}$ is generally finite.

Again, we have seen [cf. (50), (51)] that the vector $\frac{1}{2}\operatorname{curl}\mathbf{D}$ or, in the present notation, $\frac{1}{2}\,dt\,.\operatorname{curl}\mathbf{v}$ is the rotation of a particle. Hence, the rotation per unit time or the *angular velocity*, or **vortex velocity**,* will be given by the vector

$$\mathbf{w} = \tfrac{1}{2}\operatorname{curl}\mathbf{v}. \tag{76}$$

The *cubic dilatation* will be in the present notation, by (59), in time dt:

$$dt\,.\operatorname{div}\mathbf{v},$$

hence, per unit time,

$$\operatorname{div}\mathbf{v}.$$

The rate of individual change of the volume $d\tau$ of a given element of the body will be

$$\frac{d}{dt}(d\tau) = \operatorname{div}\mathbf{v}\,.\,d\tau, \tag{77}$$

and *the equation of continuity*, by (60),

$$\frac{d\rho}{dt} + \rho\operatorname{div}\mathbf{v} = 0. \tag{78}$$

* In English it is commonly called '*molecular rotation*,' in German '*Wirbelgeschwindigkeit*.' I do not call it here by the usual English name 'molecular rotation,' because, first, it is not a rotation but a velocity of rotation, and, second, it has nothing molecular about it.

This is the direct expression of the invariability of the individual mass $dm = \rho\,d\tau$ of an element during its motion and deformation; for, by (77), it reduces to

$$0 = d\tau\frac{d\rho}{dt} + \rho\frac{d(d\tau)}{dt} = \frac{d}{dt}(\rho\,d\tau)$$

or to

$$\frac{d}{dt}(dm) = 0.$$

Instead of (78) we may write also, by (74*b*),

$$\frac{\partial\rho}{\partial t} + \mathbf{v}\nabla\rho + \rho\operatorname{div}\mathbf{v} = 0;$$

but the sum of the second and third term is easily shown to be $= \operatorname{div}(\rho\mathbf{v})$; hence, as a second form of the equation of continuity,

$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\mathbf{v}) = 0. \tag{78a}$$

The interpretation of this form is obvious: the (local) increment of mass in a volume-element of space is equal to the algebraic sum of the masses flowing in and out across its boundary.

If $\operatorname{curl}\mathbf{v} = 0$, the motion is called *irrotational*, or purely *dilatational*. Sometimes, especially if we are concerned with oscillations, it is also called 'longitudinal,' as was the corresponding strain in the preceding sections.

On the other hand, if $\operatorname{div}\mathbf{v} = 0$, the motion is called *solenoidal* or *circuital*, or also 'transversal.'

As with the strains above, so the most general motion of a non-rigid body may now be represented as the superposition of an irrotational and of a circuital or solenoidal motion.

In the case of irrotational motion the so-called **velocity-potential** exists, *i.e.* a scalar function ϕ of position and time, such that

$$\mathbf{v} = \nabla\phi.$$

This potential may be single valued or many valued; the first *is* always the case if the region of irrotationality be acyclic, the second *may* be the case if this region be cyclic.

On the other hand, in the case of circuital motion we may put always

$$\mathbf{v} = \operatorname{curl}\mathbf{B},$$

with the supplementary restriction

$$\operatorname{div}\mathbf{B} = 0.$$

The vector $\mathbf{B}$, playing in this case the part of the scalar ϕ, is called **the vector-potential of velocity.**

The reader may compare this with what has been said about the corresponding potentials of displacement. The interpretation, promised on that occasion, of the formulae

$$\mathbf{B}=\frac{1}{4\pi}\int\frac{\operatorname{curl}\mathbf{v}}{r}\,d\tau=\frac{1}{2\pi}\int\frac{\mathbf{w}}{r}\,d\tau$$

and

$$\phi=-\frac{1}{4\pi}\int\frac{\operatorname{div}\mathbf{v}}{r}\,d\tau$$

will be given later on, in the chapter on *Hydrodynamics* or rather in its purely kinematical part. There we shall say also a few words about 'lines of flow' and 'vortex lines.'

For the present these general remarks about the kinematics of a non-rigid body (whether it be a fluid or, more particularly, a liquid or an elastic solid) will suffice. But before passing finally to dynamical considerations it will be useful once more to consider surfaces of discontinuity, this time from the kinematical point of view.

Kinematics of Surfaces of Discontinuity.

Let again $f=0$ be the equation of a surface of discontinuity σ. If $\mathbf{D}$ be the displacement, we have, for a discontinuity *of the first order*,

$$[\mathbf{D}]=0,$$

while the vectors ∇D_1, ∇D_2, ∇D_3 and $d\mathbf{D}/dt=\mathbf{v}$ are, generally, discontinuous:

$$[\nabla D_1],\quad [\nabla D_2],\quad [\nabla D_3],\quad [\mathbf{v}]\neq 0.$$

We have already seen that the first three of these discontinuities are completely defined by a single vector $\mathbf{m}$. In order to determine the fourth discontinuity, *i.e.* that of the velocity $\mathbf{v}$, another vector must be given, say

$$[\mathbf{v}]=\mathbf{m}'.$$

Thus, the discontinuity of the first order will be defined entirely by two vectors, $\mathbf{m}$ and $\mathbf{m}'$, which, of course, may be different for different points of the surface σ.

These two vectors, however, are not mutually independent. That is to say, if the discontinuity, distributed over some surface at the instant t, is to exist also at the instant $t+dt$ on some surface, *i.e.* if it is required that the surface σ, while moving and changing its form, *should not split into two or more distinct surfaces and that it*

should not dissolve, then between the jumps $[\nabla D_1]$, etc., on the one side and $[\mathbf{v}]$ on the other side there must subsist certain relations which Hadamard, following the example of Hugoniot, calls the **kinematical conditions of compatibility.**

For discontinuities of the first order, these have been already given (in 1878) by E. B. Christoffel, who called them *the phoronomic conditions.**

These will appear in what follows in the form of a single-vector equation.

To obtain it, remember that $[\mathbf{D}]=0$, *i.e.* also

$$[D_1]=[D_2]=[D_3]=0.$$

Let us therefore consider again a scalar function ϕ of position and time, such that, on traversing the surface σ or $f=f(a_1, a_2, a_3, t)=0$,

$$[\phi]=0.$$

If this condition is to be satisfied not only at the instant t, but also at $t+dt$, we have

$$d[\phi]=0,$$

i.e.

$$1[\nabla\phi]+\left[\frac{d\phi}{dt}\right]dt=0, \qquad (a)$$

where 1 is any line-element satisfying the condition

$$1\nabla f+\frac{df}{dt}\,dt=0. \qquad (b)$$

Here we have written for the *partial* derivative with respect to the time the symbol $\frac{d}{dt}$, since $\mathbf{a}$ or a_1, a_2, a_3 have to be kept constant on differentiating with respect to t, *i.e.* we have to follow *the same particle* of the body; and it is for such derivatives that the symbol $\frac{d}{dt}$ has already been used above.

Now, take in the eq. (a) instead of ϕ, say, the first component of displacement D_1 and remember that, by the 'identical conditions,'

$$[\nabla D_1]=m_1\mathbf{n}; \qquad (71')$$

thus

$$m_1 1\mathbf{n}+\left[\frac{dD_1}{dt}\right]dt=0,$$

or, writing again $\mathbf{n}=\nabla f:\partial f/\partial n$,

$$\frac{m_1}{\partial f/\partial n}1\nabla f+\left[\frac{dD_1}{dt}\right]dt=0;$$

* See Hadamard or Zémplen, *loc. cit.*

hence, by (*h*),

$$[v_1]=\left[\frac{dD_1}{dt}\right]=m_1\frac{df}{dt}:\frac{\partial f}{\partial n}.$$

Similarly, we shall have

$$[v_2]=m_2\frac{df}{dt}:\frac{\partial f}{\partial n},\quad [v_3]=m_3\frac{df}{dt}:\frac{\partial f}{\partial n};$$

hence

$$\mathbf{m}'=[\mathbf{v}]=\mathbf{m}\frac{df}{dt}:\frac{\partial f}{\partial n}. \tag{79}$$

This is the relation between the vectors $\mathbf{m}'$ and $\mathbf{m}$, alluded to. Remember that $\frac{\partial f}{\partial n}$ is the same thing as the tensor of ∇f or, in Cartesians,

$$\frac{\partial f}{\partial n}=+\sqrt{\left(\frac{\partial f}{\partial a_1}\right)^2+\left(\frac{\partial f}{\partial a_2}\right)^2+\left(\frac{\partial f}{\partial a_3}\right)^2}.$$

A discontinuity is called **stationary**, if the surface σ, though moving about in space and changing its form, is composed always *of the same material particles, i.e.* if f is a function of a_1, a_2, a_3 *not containing the time t*,—which we may write shortly

$$f=f(a)=0.$$

If, on the other hand, *f does* contain the time t, or if

$$f=f(t,\,a)=0,$$

then the discontinuity is *not attached* to the same particles of the body. In this case we shall say, following Hadamard, that the discontinuity constitutes a **wave** (properly so called) and that it is **propagated** in the body or material medium.

Now, for a stationary discontinuity, we have, by its definition,

$$\frac{df}{dt}=0,$$

i.e. denoting now by ∇f the gradient of f in 'the space $\mathbf{r}$' (and not $\mathbf{a}$),

$$\mathbf{v}\nabla f+\frac{\partial f}{\partial t}=0.$$

This is true for each side of the surface; thus,

$$\mathbf{v}'\nabla f+\frac{\partial f}{\partial t}=0,\quad \mathbf{v}''\nabla f+\frac{\partial f}{\partial t}=0;$$

hence, by subtraction, $[\mathbf{v}]\nabla f=0$, or $[\mathbf{v}]\mathbf{n}=0$, or, finally:

$$[\mathbf{vn}]=0. \tag{80}$$

Thus, for a *stationary* discontinuity, the *normal* component of velocity remains *continuous*.

In fact, such a discontinuity being (by definition) attached always to the same particles, the body would be torn asunder unless $[\mathbf{vn}]=0$.

But if the discontinuity is not stationary or, according to the above nomenclature, if it constitutes a wave, then we may have

$$[\mathbf{vn}]\neq 0.$$

For, in this case the discontinuity does not stick to the same particles but is transferred to others and others, and thus does not produce a tearing of the body.

Such a wave is propagated in the body or medium (which may itself move). Let σ and σ' (Fig. 41) represent the wave at the

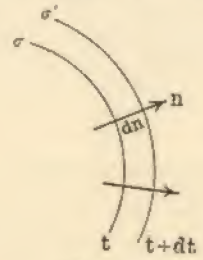

FIG. 41.

instants t and $t+dt$, respectively. If dn is an infinitesimal segment of the normal, contained between σ and σ', **the velocity of propagation** of the wave, say $\mathfrak{v}$, will be

$$\mathfrak{v}=\frac{dn}{dt}.$$

Now, in the equation (*h*) take, instead of $\mathbf{l}$, the infinitesimal normal vector, *i.e.* $\mathbf{n}\,dn$; then

$$\mathbf{n}\nabla f.\,dn+\frac{df}{dt}\,dt=0,$$

or, writing **again** $\nabla f=\mathbf{n}\,\partial f/\partial n$ and remembering that $\mathbf{n}^2=1$,

$$\frac{\partial f}{\partial n}\,dn+\frac{df}{dt}\,dt=0;$$

hence

$$\mathfrak{v}=-\frac{df}{dt}:\frac{\partial f}{\partial n}. \tag{81}$$

Finally, substituting this in (79), we have the required *kinematical condition of compatibility*:

$$[\mathbf{v}]=\left[\frac{d\mathbf{D}}{dt}\right]=-\mathfrak{v}\mathbf{m}. \tag{82}$$

To resume, write again the identical conditions (71′) or, in the condensed shape (71),

$$[\nabla(\mathbf{Dx})]=\mathbf{n}(\mathbf{mx}), \qquad (71\,bis)$$

where $\mathbf{x}$ is an arbitrary vector.

Then it is seen that the discontinuity of the first order is completely defined by *one vector* $\mathbf{m}$ *and by one scalar* $\mathfrak{v}$, the velocity of propagation.* Both the vector and the scalar may be prescribed for each point of the wave in a quite arbitrary manner. Thus, to find the velocity of propagation we must know, besides the kinematics, something else about the deformable body, for instance its dynamics. A concrete example of such a determination of the velocity of propagation, originally due to Hugoniot, will be reproduced vectorially at the very end of the chapter on Hydrodynamics (Chap. VI.). Meanwhile let us look for the main features of the Dynamics of any non-rigid body.

Stress. Differential Equations of Motion.

Let us imagine anywhere in a deformable body or medium a surface-element $d\sigma$, of which the normal may be given by the unit vector $\mathbf{n}$.

Let the **pressure** on $d\sigma$, as regards intensity and direction, be represented by the vector

$$\mathbf{p}_n\,d\sigma,$$

or, per unit area, by the vector $\mathbf{p}_n$, where n is a simple index reminding us that an element normal to $\mathbf{n}$ is in question (Fig. 42).

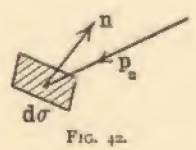

FIG. 42.

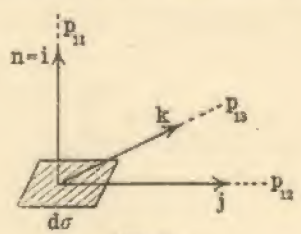

FIG. 43.

Tensions will be considered as negative pressures.

The pressures for $\mathbf{n}=\mathbf{i}, \mathbf{j}, \mathbf{k}$ will be denoted by the vectors $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, respectively, and their components along $\mathbf{i}, \mathbf{j}, \mathbf{k}$ by the scalars

* The same thing might be shown to be true also for a discontinuity of any higher order; but, as has been said above, we shall limit ourselves to those of the first order.

$p_{11}, p_{12}, \cdots p_{32}, p_{33}$, the components with equal indices being, thus, *normal* and those with different indices *tangential* to the surface acted on (Fig. 43).

For any given direction of $\mathbf{n}$, the *normal* component of the pressure will be

$$p_{nn}=\mathbf{np}_n,$$

and the *tangential* part of the pressure, in intensity and direction,

$$\mathbf{p}_n-\mathbf{n}(\mathbf{np}_n).$$

Finally, the components of $\mathbf{p}_n$ along the conventional $\mathbf{i}, \mathbf{j}, \mathbf{k}$ will, according to the above, be denoted by

$$\mathbf{ip}_n=p_{n1}, \quad \mathbf{jp}_n=p_{n2}, \quad \mathbf{kp}_n=p_{n3}.$$

The **stress** is determined completely, if the vector $\mathbf{p}_n$ be known for every point of the body or medium and for every orientation of $d\sigma$ or, what is the same thing, for every direction of $\mathbf{n}$.

Now, assuming that the resultant force of the stress, on an element of volume, is a magnitude of *the same order* as (or at least not lower than) this volume itself, it is easily proved that

$$\left.\begin{aligned} p_{n1}&=p_{11}n_1+p_{21}n_2+p_{31}n_3,\\ p_{n2}&=p_{12}n_1+p_{22}n_2+p_{32}n_3,\\ p_{n3}&=p_{13}n_1+p_{23}n_2+p_{33}n_3, \end{aligned}\right\} \qquad (83)$$

where n_1, n_2, n_3 are the components of the unit vector $\mathbf{n}$ along $\mathbf{i}, \mathbf{j}, \mathbf{k}$, or the direction cosines of the normal $\mathbf{n}$ relatively to the same system of axes.

Thus, in the most general case, the stress will be determined by nine scalars, say:

$$\begin{matrix} p_{11} & p_{12} & p_{13},\\ p_{21} & p_{22} & p_{23},\\ p_{31} & p_{32} & p_{33}. \end{matrix}$$

(Also, it will next be shown that, for dynamical reasons, say, by d'Alembert's Principle, this number is at once reduced to *six*.)

According to (83) it may be said that the pressure $\mathbf{p}_n$ is a linear vector function of the normal $\mathbf{n}$ of the surface-element acted on, which fact may be expressed shortly, denoting by p a linear vector-operator, **the stress-operator**, and writing

$$\mathbf{p}_n=p\mathbf{n} \qquad (84)$$

in the same way as, in the general theory of such operators, $\mathbf{B}=\omega\mathbf{A}$.

Thus, the theory of stress is reduced at once to the theory of the linear vector-operator; hence it would be superfluous to dwell

more on this subject. It will be sufficient to add a few words as to the terminology. Thus, **principal axes of a stress** are called those directions $\pm\mathbf{n}$, for which the pressure becomes purely normal; hence, the principal stress-axes and the corresponding **principal pressures** (or tensions) are synonyms of the principal axes and the principal values of the linear vector-operator p. If, in particular, this operator degenerates into a simple scalar, then the pressure is purely normal and equal for all directions of $\mathbf{n}$, or becomes what is called a '*hydrostatic*' pressure, *i.e.* an *isotropic* pressure.

Taking p_{32} instead of p_{23}, etc., another stress is obtained which is called **conjugate** with respect to that given by the operator p; let it be denoted by q, *i.e.*

$$q_{\iota\kappa}=p_{\kappa\iota}\quad(\iota,\ \kappa=1,\ 2,\ 3).$$

Then (83) may be written thus:

$$p_{n1}=\mathbf{n}\mathfrak{q}_1,\quad p_{n2}=\mathbf{n}\mathfrak{q}_2,\quad p_{n3}=\mathbf{n}\mathfrak{q}_3 \tag{85'}$$

or, in a single vector equation,

$$\mathbf{p}_n=p\mathbf{n}=\mathbf{i}\,.\,\mathfrak{q}_1\mathbf{n}+\mathbf{j}\,.\,\mathfrak{q}_2\mathbf{n}+\mathbf{k}\,.\,\mathfrak{q}_3\mathbf{n}, \tag{85}$$

so that the operator p, defining the stress completely, assumes again the form of a *dyadic*, as mentioned before, namely

$$p=\mathbf{i}\,.\,\mathfrak{q}_1+\mathbf{j}\,.\,\mathfrak{q}_2+\mathbf{k}\,.\,\mathfrak{q}_3.$$

Now, let $d\tau$ be a volume-element of the deformable body, $d\sigma$ a surface-element of its boundary, with normal $\mathbf{n}$ directed outwards, ρ the density of mass, and, finally, $\mathbf{F}$ the impressed (or external) force, *per unit volume*.

Among the virtual displacements of the body, considered as a system free to move about in space (*i.e.* relatively to our original system of reference O), are included:

1° the displacement of the whole body in any direction whatever, as if it were a rigid body;

2° the rotation of the body as a whole about any axis.

Thus, applying d'Alembert's Principle (1), Chap. II., first to the displacement 1° and then to 2°, we have, respectively,

$$\int\rho\ddot{\mathbf{r}}\,d\tau=\int\rho\mathbf{F}\,d\tau-\int\mathbf{p}_n d\sigma, \tag{a}$$

$$\int\rho\mathrm{V}\mathbf{r}\ddot{\mathbf{r}}\,d\tau=\int\rho\mathrm{V}\mathbf{r}\mathbf{F}\,d\tau-\int\mathrm{V}\mathbf{r}\mathbf{p}_n d\sigma, \tag{b}$$

the integrals extending to the whole volume and over the whole

bounding surface of the body, and $\ddot{\mathbf{r}}$ being a short symbol for $d^2\mathbf{r}/dt^2$ or $d^2\mathbf{D}/dt^2$ or, finally, $d\mathbf{v}/dt$.

The first of these equations expresses the principle of the centre of mass and the second the principle of areas. Both may be applied also to any portion τ' of the body if only instead of σ the surface σ', the boundary of τ', is taken and if by $\mathbf{p}_n$ the pressures on σ' are meant. These 'pressures' are the expression of the

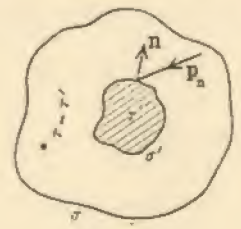

FIG. 44.

physical connexion of the portion τ' with the rest of the body $\tau-\tau'$ (Fig. 44).

We shall preserve in (*a*), (*b*) the symbols $d\tau$, $d\sigma$ (without accents), but shall keep in mind that (*a*), (*b*) are applicable to *any portion* τ' of the body.

Now, consider the component along $\mathbf{i}$ of the integral $\int\mathbf{p}_n d\sigma$ in (*a*). By (85′) it may be written

$$\int p_{n1}d\sigma=\int\mathfrak{q}_1\mathbf{n}\,d\sigma=\int\mathrm{div}\,\mathfrak{q}_1\,d\tau,$$

by Theorem VI. of Chapter I. Similarly, the two remaining components will be

$$\int p_{n2}d\sigma=\int\mathrm{div}\,\mathfrak{q}_2 d\tau,\quad\int p_{n3}d\sigma=\int\mathrm{div}\,\mathfrak{q}_3 d\tau;$$

hence, as (*a*) is applicable separately to every element of volume, $d\tau$, of the body,

$$\rho\frac{d^2\mathbf{r}}{dt^2}=\rho\mathbf{F}-\mathbf{i}\,\mathrm{div}\,\mathfrak{q}_1-\mathbf{j}\,\mathrm{div}\,\mathfrak{q}_2-\mathbf{k}\,\mathrm{div}\,\mathfrak{q}_3. \tag{86}$$

This general equation of motion of a deformable body contains the stress q conjugate to p. But, using (*b*), it may immediately be shown that the stress p must be *self-conjugate*, *i.e.* that $q=p$.

In fact, substituting (86) for $\rho\ddot{\mathbf{r}}$ in (*b*), we have

$$\int \mathrm{V}\mathbf{r}(\mathbf{i}\,\mathrm{div}\,\mathbf{q}_1 + \ldots)\,d\tau = \int \mathrm{V}\mathbf{r}\mathbf{p}_n\,d\sigma$$

$$= \int \mathrm{V}\mathbf{r}(\mathbf{i}\,.\,\mathbf{q}_1\mathbf{n} + \mathbf{j}\,.\,\mathbf{q}_2\mathbf{n} + \mathbf{k}\,.\,\mathbf{q}_3\mathbf{n})\,d\sigma, \text{ by (85)};$$

but here the volume-integral on the right hand may be easily transformed * into the surface-integral on the left hand *plus* the volume-integral

$$I = \int \{(\mathbf{q}_1\nabla)\mathrm{V}\mathbf{i}\mathbf{r} + (\mathbf{q}_2\nabla)\mathrm{V}\mathbf{j}\mathbf{r} + (\mathbf{q}_3\nabla)\mathrm{V}\mathbf{k}\mathbf{r}\}\,d\tau.$$

Hence $I = 0$; and since this is true for any portion of the body, we have everywhere

$$(\mathbf{q}_1\nabla)\mathrm{V}\mathbf{i}\mathbf{r} + (\mathbf{q}_2\nabla)\mathrm{V}\mathbf{j}\mathbf{r} + (\mathbf{q}_3\nabla)\mathrm{V}\mathbf{k}\mathbf{r} = 0. \qquad (c)$$

Multiply this equation scalarly by $\mathbf{i}$ and remember that $\mathbf{i}\mathrm{V}\mathbf{i}\mathbf{r} = 0$, $\mathbf{i}\mathrm{V}\mathbf{j}\mathbf{r} = \mathbf{r}\mathrm{V}\mathbf{i}\mathbf{j} = \mathbf{r}\mathbf{k} = r_3$ and, similarly, $\mathbf{i}\mathrm{V}\mathbf{k}\mathbf{r} = -r_2$; then

$$(\mathbf{q}_2\nabla)r_3 - (\mathbf{q}_3\nabla)r_2 = 0;$$

but $\mathbf{q}_2\nabla = q_{21}\dfrac{\partial}{\partial r_1} + q_{22}\dfrac{\partial}{\partial r_2} + q_{23}\dfrac{\partial}{\partial r_3}$, $\mathbf{q}_3\nabla = q_{31}\dfrac{\partial}{\partial r_1} + q_{32}\dfrac{\partial}{\partial r_2} + q_{33}\dfrac{\partial}{\partial r_3}$;

$$\therefore\ q_{23} = q_{32} = p_{23} = p_{32};$$

similarly, multiplying (*c*) scalarly by $\mathbf{j}$, or by $\mathbf{k}$, we get

$$q_{31} = q_{13} = p_{31} = p_{13}, \quad q_{12} = q_{21} = p_{12} = p_{21}.$$

Hence $q = p$. Q.E.D.

Thus, the stress is *self-conjugate*, or, as is also said, *irrotational*;† in other words, the corresponding linear operator p is a *symmetrical* operator. The number of scalars defining it is reduced to *six*, say:

p_{11}, p_{22}, p_{33} (normal pressures),

p_{23}, p_{31}, p_{12} (tangential pressures).

This property, as we have seen, belongs to *every dynamical stress*, *i.e.* to every stress obeying d'Alembert's (or any other equivalent) Principle.

* For this purpose it suffices to recur to the formula

$$\mathrm{div}\,(s\mathbf{A}) = s\,\mathrm{div}\,\mathbf{A} + \mathbf{A}\nabla s,$$

which is satisfied identically for any scalar s and any vector $\mathbf{A}$, and which gives

$$\int s\,\mathrm{div}\,\mathbf{A}\,.\,d\tau = \int \mathrm{div}\,(s\mathbf{A})\,.\,d\tau - \int \mathbf{A}\nabla s\,.\,d\tau = \int s\,\mathbf{A}\mathbf{n}\,d\sigma - \int \mathbf{A}\nabla s\,.\,d\tau.$$

† This name is based on the circumstance that the differences

$$p_{32} - p_{23}, \text{ etc.},$$

would be the components of the moment of a couple of forces (per unit volume) tending to *turn* an element $d\tau$; and since $p_{32} = p_{23}$, etc., this moment vanishes.

It may, by the way, be remarked here that Maxwell's 'electromagnetic stress' (for isotropic bodies *is*, but) for *crystalline* bodies *is not* irrotational and, consequently, does not fulfil the above mechanical requirements.

Henceforth, by the last proposition, we may write in all our formulae, p instead of q. Thus, the general differential equation of motion (86) becomes

$$\rho\frac{d\mathbf{v}}{dt} = \rho\frac{d^2\mathbf{r}}{dt^2} = \rho\mathbf{F} - \mathbf{i}\,\mathrm{div}\,\mathbf{p}_1 - \mathbf{j}\,\mathrm{div}\,\mathbf{p}_2 - \mathbf{k}\,\mathrm{div}\,\mathbf{p}_3. \qquad (87)$$

This is the vectorial condensation of the three commonly used scalar equations, of which it will suffice here to write the first only:

$$\rho\frac{dv_1}{dt} = \rho F_1 - \mathrm{div}\,\mathbf{p}_1 = \rho F_1 - \left(\frac{\partial p_{11}}{\partial r_1} + \frac{\partial p_{12}}{\partial r_2} + \frac{\partial p_{13}}{\partial r_3}\right), \qquad (87a)$$

r_1, r_2, r_3 being the same as the ordinary 'x, y, z.'

For a **non-viscous fluid** the tangential pressures vanish and those normal become equal for all directions of $\mathbf{n}$, that is to say:

$$p_{23} = p_{31} = p_{12} = 0$$

and

$$p_{11} = p_{22} = p_{33}, \text{ say} = p.$$

The linear operator, which we have in the general case denoted also by p, becomes now a simple scalar, *i.e.* a scalar function of position and of time. In this simplest case eq. (87*a*) reduces to

$$\rho\frac{d^2r_1}{dt^2} = \rho\frac{dv_1}{dt} = \rho F_1 - \nabla_1 p,$$

and the vector equation (87) becomes

$$\rho\frac{d^2\mathbf{r}}{dt^2} = \rho\frac{d\mathbf{v}}{dt} = \rho\mathbf{F} - \nabla p. \qquad (88)$$

To this fundamental equation of *Hydrodynamics* we shall return in the next chapter.

For **viscous** liquids or (more generally) fluids, and for **elastic solids** the stress loses this idyllic simplicity; the pressure (or tension) is no longer isotropic and, in general, not normal, but becomes normal for three particular directions only—the principal axes, varying from point to point. In this case we must return to the general equation of motion, (87), taking into account the six different components of stress

$$p_{11}, p_{22}, p_{33}, p_{23}, p_{31}, p_{12}$$

which depend generally in a rather complicated way on the elementary deformations of the body.

We shall not enter here into the details of this subject, but shall limit ourselves to the brief remark that, for infinitesimal deformations, the normal and the tangential components of stress have a *potential*, *i.e.* may be represented as the partial derivatives of a scalar function f with respect to the six quantities which determine the strain.

That is to say, denoting again by **D** the displacement and by x_{11}, etc., x_{23}, etc., the principal linear elongations and the shears,

$$\nabla_i D_i = x_{ii},$$

$$\nabla_i D_k + \nabla_k D_i = x_{ik} = x_{ki},$$

we have

$$\left.\begin{aligned} p_{ii} &= \frac{\partial f}{\partial x_{ii}}; \quad p_{ik} = \frac{\partial f}{\partial x_{ik}} = \frac{\partial f}{\partial x_{ki}} = p_{ki}, \\ i,\ k &= 1,\ 2,\ 3, \end{aligned}\right\} \tag{89}$$

where f is a quadratic homogeneous function of the six 'strain-characteristics' x_{ii}, x_{ik}.

This function, taken with the *negative* sign, constitutes the so-called **energy of elastic deformation**, per unit volume.

Remembering that the virtual work of the pressures or tensions will then be given, per unit volume, by δf, the differential equations of motion of an elastic body may be easily deduced from Hamilton's Principle.

Finally, it may be observed here that for non-viscous fluids the energy of deformation depends only on the cubic dilatation, *i.e.* the above function f reduces to

$$f = f(\theta) = f(x_{11} + x_{22} + x_{33}),$$

whence it follows immediately that

$$p_{23} = p_{31} = p_{12} = 0$$

and

$$p_{11} = p_{22} = p_{33},$$

as stated above.

CHAPTER VI.

HYDRODYNAMICS.

Fundamental Notions and Equations.

In the case of a non-viscous fluid, of which the dynamics shall be the subject of the present chapter, the stress, as has been mentioned above, reduces to a pressure p purely normal and isotropic, not only in the state of equilibrium, but also in any state of motion.

Under these conditions the equation (88) holds, where $\mathbf{v}$ is the velocity, ρ the density of the fluid, **F** the impressed force, per unit volume. We shall limit ourselves to the consideration of the so-called *Eulerian* form* of the hydrodynamical equations, and shall, consequently, choose as independent variables the time t and any three coordinates which determine the position of a point in space relatively to some system of reference which does not participate in the considered motion of the fluid, and which is mechanically equivalent to our previous 'system O.'

Thus, understanding by ∇ the gradient in this space and using the relation (74*a*), we have, as the fundamental differential equation of Hydrodynamics,

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = \mathbf{F} - \frac{1}{\rho}\nabla p. \tag{90}$$

In addition, we have, by (78) and (78*a*), *the equation of continuity*:

$$\frac{d\rho}{dt} + \rho \operatorname{div} \mathbf{v} = \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0. \tag{91}$$

If then the relation between the pressure and the density

$$\mathfrak{F}(\rho,\ p) = 0, \tag{92}$$

* As distinguished from the so-called *Lagrangian* form, which will be omitted here, and in which a_1, a_2, a_3, t are the independent variables.

characteristic of the fluid considered, be given, we have three equations: one vectorial and two scalar, for one vector $\mathbf{v}$ and for the two scalars p, ρ. Thus, the initial data $\mathbf{v}_0$, ρ_0 and the corresponding p_0, together with eventual data for the bounding surfaces, will completely determine the course of phenomena in time; the forces $\mathbf{F}$ being always considered as given for every point and for all times.

It may be observed here that, in the more general case, the relation between density and pressure may contain also the temperature; in this case, of course, the above equations do not constitute by themselves a complete system. But we shall suppose that the relation (92) *contains* p, ρ *only* (with a single exception to be considered later, in a section on vortex motion).

If the fluid be, in particular, an incompressible liquid, we have simply, as a special case of (92), $\rho = const.$ and, consequently, $\operatorname{div}\mathbf{v} = 0$. But in general we shall admit compressibility.

The **free surface** forming the boundary (or a part of the boundary) of the fluid in motion, if it exists, is easily shown to be composed always of the same material particles; hence, if its equation be $f = 0$, we have

$$\frac{df}{dt} = 0$$

or, by what has been explained previously,

$$\frac{\partial f}{\partial t} + \mathbf{v}\nabla f = 0.$$

In all the points of a rigid, fixed wall, bounding the fluid, completely or partially, the velocity $\mathbf{v}$ is tangent to the wall, so that

$$\mathbf{vn} = 0,$$

$\mathbf{n}$ being the (say, unit) vector normal to the wall.

If a rigid body, being immersed in the fluid, moves about in it, then, at any point of its surface, $\mathbf{vn}$ is equal to the normal component of the velocity of the rigid body at the given point.

The differential equation to a **line of flow**, *i.e.* a line having at every point the direction of $\mathbf{v}$, will be, in vector form,

$$d\mathbf{s} = \lambda\mathbf{v}\,ds,$$

where $d\mathbf{s}$ is any element of the line, regarded as a vector, and λ a scalar.

The differential equation of a **stream line** or of the path of an individual particle is simply

$$\frac{d\mathbf{r}}{dt} = \mathbf{v},$$

where $\mathbf{r}$ is to be considered as an unknown function of the time.

The stream lines coincide with the lines of flow when and only when the motion of the fluid is **steady**, or locally invariable, *i.e.* if $\partial\mathbf{v}/\partial t = 0$. But in the general case they must be carefully distinguished.

The vortex velocity ('molecular rotation') will be denoted, as before, by $\mathbf{w}$; thus

$$\mathbf{w} = \tfrac{1}{2}\operatorname{curl}\mathbf{v}.$$

Whence the equation of a **vortex line** (having everywhere the direction of the vector $\mathbf{w}$),

$$d\mathbf{s} = \lambda\mathbf{w}\,ds,$$

the meaning of the symbols being as above. A tubular portion of the fluid bounded by a surface generated by vortex lines is called a **vortex tube** (Fig. 45), and in particular, if it be of infinitesimal

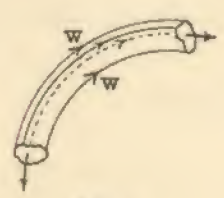

FIG. 45.

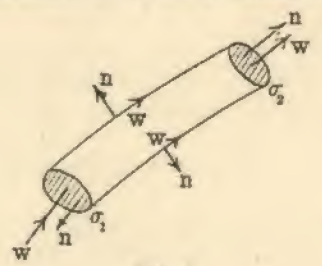

FIG. 46.

section, a **vortex filament**. The product of the cross section into the vortex velocity, taken scalarly, is called the **moment** or the **strength** of a vortex filament. Any vortex tube may be decomposed (mentally) into filaments, and thus its moment will be the sum, or the integral, of the elementary moments of all filaments.

Since $$\nabla\mathbf{V}\nabla\mathbf{v} = 0$$

or $$\operatorname{div}\operatorname{curl}\mathbf{v} = 0,$$

identically, we have $\operatorname{div}\mathbf{w} = 0$, and consequently, for any closed surface σ,

$$\int \mathbf{wn}\,d\sigma = 0,$$

$\mathbf{n}$ being the surface-normal. Applying this to a portion of a vortex tube contained between two cross sections σ_1, σ_2 (Fig. 46),

and remembering that on the lateral surface of the tube $\mathbf{wn}=0$, it is at once seen that

the moment of a vortex tube is the same throughout its length (*i.e.* $w_1\sigma_1 = w_2\sigma_2$).

This is a purely kinematical property. The invariability of the moment of a vortex in time will be seen, later on, to follow from the dynamical equations.

It is often useful to consider the line-integral of the velocity $\mathbf{v}$ along a curve s drawn in the fluid; if this curve be a closed one, or a **circuit**, the line-integral, say I,

$$I = \int_{(s)} \mathbf{v}\, d\mathbf{s},$$

is called the **circulation** round the curve. By $\frac{dI}{dt}$ we shall denote, later on, the time rate of change of the circulation round a circuit *composed always of the same particles of fluid*, in agreement with what was said before (Chap. V.) in regard to the meaning of the symbol $\frac{d}{dt}$.

It will be useful to recall here the Theorem of Stokes (see Chapter I.), according to which

$$I = \int_{(s)} \mathbf{v}\, d\mathbf{s} = \int \mathbf{n}\,\mathrm{curl}\,\mathbf{v}\,.\,d\sigma = 2\int \mathbf{wn}\, d\sigma, \qquad \text{(XVIII.)}$$

σ being any surface limited by the circuit s. Remember that for a person looking along the direction of the surface-normal $\mathbf{n}$ the positive way of integration round s is clockwise.

From (XVIII.) we see at once that the circulation I round all circuits embracing a vortex tube once only, say in the positive sense, *has one and the same value*, namely equal to twice the surface-integral $\int \mathbf{wn}\, d\sigma$, *i.e.* to *twice the moment of the vortex tube.*

For a circuit embracing a vortex tube N times, we have

$$I = 2N \times \text{moment of vortex tube},$$

since such a circuit may be substituted by N circuits, every one of which embraces the vortex tube once only. The case of a circuit s embracing a vortex ring twice and the splitting of s into two simple

circuits, s_1, s_2, is shown on Fig. 47. This splitting or reduction of s is accomplished by introducing an auxiliary path, such as AB;

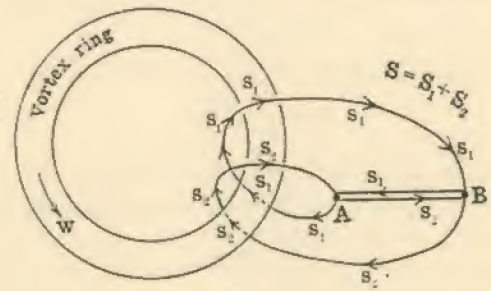

FIG. 47.

and the equivalence of the two simple circuits to the given circuit s (as regards the value of the circulation) is seen at once by noticing that $\int_A^B \mathbf{v}\, d\mathbf{s} + \int_B^A \mathbf{v}\, d\mathbf{s}$ is *nil.*

All these properties are, of course, quite independent of the dynamics of the fluid.

Having now stated these properties, let us return to the hydrodynamical equation (90). Denoting the components of any vector along $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ by the indices 1, 2, 3, we obtain easily

$$\begin{aligned}(\mathbf{v}\nabla)\mathbf{v} &= \mathbf{i}(\mathbf{v}\nabla)v_1 + \mathbf{j}(\mathbf{v}\nabla)v_2 + \mathbf{k}(\mathbf{v}\nabla)v_3\\ &= \mathbf{i}\{v_1\,.\,\nabla_1 v_1 + v_2\,.\,\nabla_2 v_1 + v_3\,.\,\nabla_3 v_1\} + \mathbf{j}\{\dots\} + \mathbf{k}\{\dots\}\\ &= \mathbf{i}\left\{\nabla_1\left(\frac{v^2}{2}\right) + v_2(\nabla_2 v_1 - \nabla_1 v_2) + v_3(\nabla_3 v_1 - \nabla_1 v_3)\right\} + \dots\\ &= \mathbf{i}\left\{\nabla_1\left(\frac{v^2}{2}\right) + 2(w_2 v_3 - w_3 v_2)\right\} + \mathbf{j}\{\dots\} + \mathbf{k}\{\dots\},\end{aligned}$$

i.e. identically, for any vector $\mathbf{v}$:

$$(\mathbf{v}\nabla)\mathbf{v} = \nabla\left(\frac{v^2}{2}\right) + 2\mathrm{V}\mathbf{wv}, \qquad \text{(XIX.)}$$

where $\mathbf{w} = \frac{1}{2}\,\mathrm{curl}\,\mathbf{v}$.

Using this formula of transformation we may write, instead of (90),

$$\frac{\partial \mathbf{v}}{\partial t} + \tfrac{1}{2}\nabla(v^2) + 2\mathrm{V}\mathbf{wv} = \mathbf{F} - \frac{1}{\rho}\nabla p. \qquad (a)$$

Henceforth we shall suppose that the impressed forces have a scalar *potential* (not necessarily single-valued), *i.e.* that

$$\mathbf{F} = -\nabla\Phi,$$

and also that the density ρ is given as a function of the *pressure* p *alone*. Under these assumptions

$$\mathbf{F} - \frac{1}{\rho}\nabla p = \nabla Q,$$

where Q is a scalar, namely

$$Q = -\Phi - \int \frac{dp}{\rho},$$

and, in particular, for an incompressible fluid, $Q = -\Phi - p/\rho$.

The differential equation of fluid motion will then become

$$\frac{d\mathbf{v}}{dt} = \nabla Q, \tag{93}$$

or, by (*a*),

$$\frac{\partial \mathbf{v}}{\partial t} + 2\mathrm{V}\boldsymbol{\omega}\mathbf{v} = \nabla(Q - \tfrac{1}{2}v^2). \tag{93a}$$

Thus, under the stated conditions, *the acceleration* $d\mathbf{v}/dt = d^2\mathbf{r}/dt^2$ *has a scalar potential* Q, so that the *curl* of $d\mathbf{v}/dt$ is permanently equal to zero.

Conservation of Energy.

Let us consider an individual portion τ of the fluid, mentally separated from the whole mass of fluid and bounded by the surface σ. If $f = 0$ be the equation to this surface, we have

$$df/dt = \partial f/\partial t + \mathbf{v}\nabla f = 0,$$

since σ itself, equally with the whole portion τ, consists always of the same fluid particles.

The kinetic energy of this portion of fluid is

$$L = \tfrac{1}{2}\int v^2 \rho \, d\tau,$$

and since $\frac{d}{dt}(\rho\, d\tau) = 0$, the change of L, per unit time, will be

$$\frac{dL}{dt} = \int \rho \mathbf{v} \frac{d\mathbf{v}}{dt} d\tau,$$

i.e., by (90),

$$\frac{dL}{dt} = \int \rho \mathbf{v}\mathbf{F} \,.\, d\tau - \int \mathbf{v}\nabla p \,.\, d\tau.$$

Now, the first integral on the right side is the work done per unit time on the portion of fluid under consideration by the impressed forces $\mathbf{F}$; let us denote it by W. Again, as to the second integral,

we have $\mathbf{v}\nabla p = \operatorname{div}(\mathbf{v}p) - p \operatorname{div}\mathbf{v}$, whence, denoting by $\mathbf{n}$ the normal of σ directed *towards* τ,

$$-\int \mathbf{v}\nabla p \,.\, d\tau = \int p\mathbf{v}\mathbf{n}\, d\sigma + \int p \operatorname{div}\mathbf{v} \,.\, d\tau$$
$$= \int p\mathbf{v}\mathbf{n}\, d\sigma + \int p \frac{d}{dt}(d\tau), \quad \text{by (77)};$$

hence, finally,

$$W + \int p\mathbf{v}\mathbf{n}\, d\sigma = \frac{dL}{dt} - \int p \frac{d}{dt}(d\tau), \tag{94}$$

which equation may be read thus:

The work of the impressed forces, *plus* the work of the pressure exerted on the surface σ, is equal to the increment of kinetic energy, *plus* the work absorbed by the fluid on its being condensed,—everything per unit time.

The last term in this equation of energy, *i.e.* the term corresponding to the 'energy of deformation' of the elements of fluid, may also be written, for every element,

$$-p \operatorname{div}\mathbf{v} \,.\, d\tau = p \frac{d \log \rho}{dt} d\tau.$$

Since the density ρ depends only on the pressure p, we have

$$p \frac{d \log \rho}{dt} = \rho \frac{du}{dt},$$

where

$$u = -\int p\, d\left(\frac{1}{\rho}\right).$$

Hence

$$W + \int p\mathbf{v}\mathbf{n}\, d\sigma = \frac{dL}{dt} + \int \frac{du}{dt} \,.\, \rho\, d\tau;$$

and since $\frac{d}{dt}(\rho\, d\tau) = 0$, we may write, finally,

$$W + \int p\mathbf{v}\mathbf{n}\, d\sigma = \frac{d}{dt}(L + U) \tag{94a}$$

where

$$U = \int u\rho\, d\tau.$$

Thus $L + U$ plays the part of *the total energy* of the fluid mass considered, L being its kinetic energy and U its 'energy of deformation'; the last named energy per unit mass is

$$u = -\int p\, d\epsilon, \tag{95}$$

$\epsilon = 1/\rho$ being the specific volume of the fluid.

Thus, the work of impressed forces together with the work of the pressure exerted on the surface σ is equal to the increment of total energy of the fluid bounded by this surface.

Before drawing special conclusions from the differential hydrodynamical equations, we shall give here, in vector form, their transformation, due to Clebsch, which is, for more than one reason, remarkable.

Clebsch's Transformation.

A vector, or rather a vector-field, of any distribution in space, and consequently the above velocity $\mathbf{v}$, may be put into the form

$$\mathbf{v} = \nabla\phi + \lambda\nabla\psi, \tag{96}$$

ϕ, λ, ψ being three, in the general case mutually independent, scalar functions of position and time. In hydrodynamics this form was used by Clebsch (1856-58), in order to effect the corresponding transformation of the differential equations of motion.* Clebsch, of course, made use of scalar language.

Writing $\text{curl} = V\nabla$ and remembering that $V\nabla . \nabla\psi$ or $\text{curl}\,\nabla\psi$ vanishes, identically, we get immediately from (96) the vortex vector $\mathbf{w} = \frac{1}{2}\,\text{curl}\,\mathbf{v}$, namely

$$\mathbf{w} = \tfrac{1}{2}V\nabla\lambda . \nabla\psi, \tag{97}$$

which is the vector equivalent of the three scalar formulae of Clebsch, $w_1 = \frac{1}{2}\left(\frac{\partial\lambda}{\partial y}\frac{\partial\psi}{\partial z} - \frac{\partial\lambda}{\partial z}\frac{\partial\psi}{\partial y}\right)$, etc.

Since the vectors $\nabla\lambda$, $\nabla\psi$ are normal to the surfaces $\lambda = \text{const.}$, $\psi = \text{const.}$, respectively, the vector $\mathbf{w}$ is tangent to both surfaces, so that by (97) we see almost immediately that

the vortex lines are the intersections of the surfaces

$$\lambda = const., \quad \psi = const.$$

If, in particular, these surfaces coincide with one another, *i.e.* if ψ is a function of λ only, we have, by (97),

$$\mathbf{w} = 0.$$

Hence, if ψ is a function of λ only, the motion is irrotational, and (as is easily seen) *vice versa*.

Now, since in the general case the fluid motion may be rotational, we shall consider λ, ψ as mutually *independent* functions (of position

* For the literature of this subject see A. B. Basset's *Treatise on Hydrodynamics*, Cambridge, 1888, Vol. I. p. 28.

and time). The third function ϕ is also, in the general case, independent of the first two.

It may be observed in passing that, by (96) and (97),

$$\mathbf{vw} = \tfrac{1}{2}\nabla\phi . V\nabla\lambda . \nabla\psi, \tag{98}$$

or $\mathbf{vw} = \frac{1}{2}$ volume of the parallelopiped constructed on the vectors $\nabla\phi$, $\nabla\lambda$, $\nabla\psi$ as edges; whence it is easily seen that the lines of flow constitute an *orthogonal* network with the vortex lines *when and only when* the surfaces $\phi = \text{const.}$, $\lambda = \text{const.}$ and $\psi = \text{const.}$ have common lines of intersection, or, in other words, when the vortex lines lie wholly on the surfaces $\phi = \text{const.}$

In order to introduce Clebsch's functions in the equation of motion (93) or (93*a*), let us take the vector product of (96) and (97). We have

$$\begin{aligned} -2V\mathbf{wv} &= V\mathbf{v}V\nabla\lambda . \nabla\psi \\ &= (\mathbf{v}\nabla\psi)\nabla\lambda - (\mathbf{v}\nabla\lambda)\nabla\psi, \text{ by (IX.)}; \end{aligned}$$

using this and (96), the equation (93*a*) becomes

$$\begin{aligned} \nabla(Q - \tfrac{1}{2}v^2) &= \nabla\left(\frac{\partial\phi}{\partial t} + \lambda\frac{\partial\psi}{\partial t}\right) + \frac{\partial\lambda}{\partial t}\nabla\psi - \frac{\partial\psi}{\partial t}\nabla\lambda \\ &\quad + (\mathbf{v}\nabla\lambda)\nabla\psi - (\mathbf{v}\nabla\psi)\nabla\lambda \\ &= \nabla\left(\frac{\partial\phi}{\partial t} + \lambda\frac{\partial\psi}{\partial t}\right) + \frac{d\lambda}{dt}\nabla\psi - \frac{d\psi}{dt}\nabla\lambda. \end{aligned}$$

Ultimately, then, the differential equation of fluid-motion is

$$\frac{d\lambda}{dt}\nabla\psi - \frac{d\psi}{dt}\nabla\lambda + \nabla H = 0, \tag{93b}$$

where the scalar H is given by

$$H = \frac{\partial\phi}{\partial t} + \lambda\frac{\partial\psi}{\partial t} + \tfrac{1}{2}v^2 - Q. \tag{99}$$

Multiplying scalarly by $\mathbf{w}$ and remembering that, by (97), $\mathbf{w}\nabla\psi = 0$, $\mathbf{w}\nabla\lambda = 0$, we get, from (93*b*),

$$\mathbf{w}\nabla H = 0$$

or $H = const.$ *along any vortex line*. And since such a line is an intersection of a surface $\lambda = \text{const.}$ with a surface $\psi = \text{const.}$, we have, in the first place,

$$H \equiv \frac{\partial\phi}{\partial t} + \lambda\frac{\partial\psi}{\partial t} + \tfrac{1}{2}v^2 - Q = H(\lambda, \psi, t).$$

But then it may be proved also that H depends neither on λ nor on ψ, and is, consequently, a function of t alone.

To see this, take the *curl* of (93*b*); then

$$\mathrm{V}\nabla\left(\frac{d\lambda}{dt}\right).\nabla\psi - \mathrm{V}\nabla\left(\frac{d\psi}{dt}\right).\nabla\lambda = 0;$$

hence, multiplying by $\nabla\lambda$ or by $\nabla\psi$, scalarly, and using (97),

$$\mathbf{w}\nabla\frac{d\lambda}{dt} = 0, \quad \mathbf{w}\nabla\frac{d\psi}{dt} = 0. \tag{a}$$

From this point we may repeat *verbatim* the reasoning of Basset (*loc. cit.*):—

A vortex line, as we know already, lies on a surface

$$\lambda = A = \text{const.};$$

but by (*a*) this line is contained at the same time in the surface

$$\lambda + \frac{d\lambda}{dt} = A;$$

now, this is possible only if $\frac{d\lambda}{dt} = 0$; similarly we shall have $\frac{d\psi}{dt} = 0$.

Thus, (93*b*) reduces to

$$\nabla H = 0, \tag{93c}$$

so that the function H *has everywhere one and the same value, which can vary only with the time*; in other terms, we have what is called a *first integral* of the differential equation of motion,

$$H \equiv \frac{\partial\phi}{\partial t} + \lambda\frac{\partial\psi}{\partial t} + \tfrac{1}{2}v^2 - Q = H(t). \tag{100}$$

At the same time we see, from the equations

$$\frac{d\lambda}{dt} = 0, \quad \frac{d\psi}{dt} = 0,$$

that *any surface* $\lambda = const.$ *or* $\psi = const.$ *is composed always of the same particles of fluid*; hence, the same thing is true for any *vortex line*, since it is the intersection of a pair of such surfaces. Thus is proved one of Helmholtz's celebrated theorems, according to which *any vortex line is composed always of the same particles.* (To this subject we shall return.)

It must be remembered that the above deductions are based on the assumption that the density is a function of the pressure only and that the impressed forces have a (scalar) potential.

Fluid in Equilibrium.

To obtain the necessary and sufficient condition for the equilibrium of a fluid, it is enough to assume simply that the initial velocity

$\mathbf{v}_0$ vanishes throughout the fluid and that for all times and for all particles $d\mathbf{v}/dt = 0$.

Then the general equation of hydrodynamics, (90), gives

$$\mathbf{F} = \frac{1}{\rho}\nabla p. \tag{101}$$

If the pressure is a function of the density only, then, putting

$$\Pi = \int\frac{dp}{\rho},$$

we have

$$\mathbf{F} = \nabla\Pi; \tag{101a}$$

hence, equilibrium is possible only if the impressed forces $\mathbf{F}$ *have a (scalar) potential*; and since p, ρ are single-valued functions of space, this potential must also be *single-valued.*

Also, by (93), putting $d\mathbf{v}/dt = 0$,

$$\nabla Q = 0,$$

whence $Q = \text{const.}$ in the whole space occupied by the fluid, which result coincides precisely with (101*a*), since with $\mathbf{F} = -\nabla\Phi$ the meaning of Q was

$$Q = -\Phi - \int\frac{dp}{\rho} = -\Phi - \Pi.$$

Neglecting an irrelevant additive constant, we may write

$$\Phi = -\Pi,$$

and as Π depends only on p, the surfaces of constant potential of impressed forces coincide with those of constant pressure, which are also surfaces of constant density.

In the absence of all impressed forces, other than pressures exerted on the bounding surface, we have, by (101*a*),

$$\nabla\Pi = 0$$

or $\Pi = \text{const.}$, and consequently

$$p = \text{const.},$$

i.e. a pressure uniform throughout the fluid and equal to that exerted on its surface (**Pascal's Law**).

If ρ *be not a function of* p *only*, we must return to the more general equation (101), which gives

$$\rho\dot{\mathbf{F}} = \nabla p,$$

whence, by curling,

$$\rho\,\mathrm{curl}\,\mathbf{F} + \mathbf{i}\left(F_3\frac{\partial\rho}{\partial y} - F_2\frac{\partial\rho}{\partial z}\right) + \mathbf{j}(\ldots) + \mathbf{k}(\ldots) = 0,$$

i.e.

$$\rho\,\mathrm{curl}\,\mathbf{F} - \mathrm{V}\mathbf{F}\nabla\rho = 0.$$

Multiplying scalarly by $\mathbf{F}$, we have

$$\mathbf{F}\,\mathrm{curl}\,\mathbf{F}=0. \tag{102}$$

Thus, in the more general case of equilibrium, in which p is not a function of ρ only, the impressed forces may have no potential; but this being the case, the equilibrium of the fluid is possible only if the 'vortex' of the impressed force-field, *i.e.* curl $\mathbf{F}$, is everywhere *normal* to the force $\mathbf{F}$ itself.

The reader will observe that (102) is but the vectorial form of the well-known condition,

$$F_1\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)+F_2\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)+F_3\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)=0,$$

necessary and sufficient in order that

$$F_1dx+F_2dy+F_3dz$$

should have an integrating factor, which in our case is the density.

Irrotational Motion.

Returning again from rest to motion, let us consider in the first place irrotational motion, *i.e.* motion free from vortices.

We shall still limit ourselves to the case in which the impressed forces have a (scalar) potential and in which the density is a function of the pressure alone. Then (93), or (93*a*), holds; hence, assuming

$$\mathbf{w}=\tfrac{1}{2}\,\mathrm{curl}\,\mathbf{v}=0, \tag{103}$$

so that $\mathbf{v}$ can be derived from a scalar potential ϕ, called the *velocity potential*, *i.e.*

$$\mathbf{v}=\nabla\phi, \tag{104}$$

we shall have, by (93*a*),

$$\frac{\partial}{\partial t}\nabla\phi=\nabla(Q-\tfrac{1}{2}v^2).$$

But the operators $\frac{\partial}{\partial t}$ and ∇ are commutative; hence

$$\nabla\left(\frac{\partial\phi}{\partial t}+\tfrac{1}{2}v^2-Q\right)=0,$$

where $v^2=(\nabla\phi)^2$.

Thus, the scalar expression in the parenthesis has a *constant* value, *i.e.* constant in space but not necessarily in time; but since an arbitrary (additive) function of the time alone is implied

already in the velocity potential,* we may simply, without impairing the generality, omit that 'constant,' and thus write

$$\frac{\partial\phi}{\partial t}+\tfrac{1}{2}v^2-Q=0 \tag{105}$$

or

$$\frac{\partial\phi}{\partial t}+\tfrac{1}{2}(\nabla\phi)^2+\Pi+\Phi=0. \tag{105a}$$

The reader will observe that this *first integral* of the equation of motion, in the case of irrotationality, would also follow immediately from the more general integral of Clebsch, (99); for, in the absence of vortices, λ is a function of ψ alone, so that putting

$$\int\lambda\,d\psi=\phi',$$

we have

$$\lambda\frac{\partial\psi}{\partial t}=\frac{\partial\phi'}{\partial t};$$

thus, to get (105) from (99), we have only to incorporate ϕ' into the velocity potential ϕ.

The equation of continuity becomes, by introducing the velocity-potential, according to (104),

$$\frac{\partial\rho}{\partial t}+\mathrm{div}\,(\rho\nabla\phi)=\frac{d\rho}{dt}+\rho\nabla^2\phi=0 \tag{106}$$

or

$$\nabla^2\phi+\frac{d\log\rho}{dt}=0. \tag{106a}$$

For an *incompressible* fluid (liquid) this becomes

$$\nabla^2\phi=0, \tag{107}$$

which is the well-known **equation of Laplace.**

If the (irrotational) motion be *steady*, (105) reduces to

$$\tfrac{1}{2}v^2+\Pi+\Phi=\mathrm{const.}, \tag{108}$$

where 'const.' *is* constant in space and in time.

For an incompressible fluid or, shortly, for a *liquid*, under the action of gravity alone,

$$\Pi=p/\rho,\quad \Phi=gh,$$

where h is the height above some level chosen conventionally, as a reference level, and g the terrestrial acceleration. Hence, by (108), for any two levels h_1, h_2, where the resultant velocities are $\mathbf{v}_1$, $\mathbf{v}_2$ and the pressures p_1, p_2, respectively,

$$h_1+\frac{p_1}{\rho g}+\frac{v_1^2}{2g}=h_2+\frac{p_2}{\rho g}+\frac{v_2^2}{2g}. \tag{108a}$$

* For, putting $\phi+F(t)$ instead of ϕ, we have still
$\mathbf{v}=\nabla\phi+\nabla F(t)=\nabla\phi+0$, as in (104).

This property is known commonly under the name of **Bernoulli's Theorem.**

We shall see, later on, that the expression

$$\tfrac{1}{2}v^2 - Q$$

has a constant value for any (not necessarily irrotational) steady motion, but constant only *along any given line of flow* and generally not throughout the whole mass of fluid. 'Bernoulli's Theorem' properly refers to *this* somewhat more general property.

Putting $p_1 = p_2$ and $v_2 = 0$, which satisfies the conditions for a liquid issuing through a small aperture from a large vessel (Fig. 48),

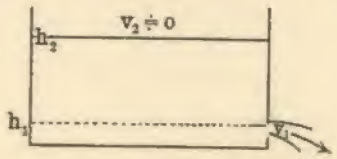

FIG. 48.

we have, by (108*a*),

$$v_1^2 = 2g(h_2 - h_1),$$

where v_1 is the velocity of issue and $h_2 - h_1$ the depth of the aperture below the free level (**Torricelli's Law**).

For compressible fluids, the equation

$$\nabla^2\phi = \operatorname{div} \mathbf{v} = -\frac{d \log \rho}{dt},$$

as has been remarked, gives

$$\phi = -\frac{1}{4\pi}\int \frac{1}{r} \operatorname{div} \mathbf{v}\, d\tau = \frac{1}{4\pi}\int \frac{1}{r}\frac{d \log \rho}{dt}\, d\tau, \qquad (109)$$

r being the distance of a volume-element $d\tau$ from the point for which the value of ϕ is to be found. This formula may be interpreted by saying that every element $d\tau$ in which the fluid density is changing contributes to the velocity-potential the elementary amount

$$\frac{d\tau}{4\pi}\frac{d \log \rho}{dt}\frac{1}{r},$$

and consequently to the velocity $\mathbf{v}$ the elementary vector quantity

$$\frac{d\tau}{4\pi}\frac{d \log \rho}{dt}\nabla\left(\frac{1}{r}\right) = -\frac{1}{r^2}\frac{d \log \rho}{dt}\frac{d\tau}{4\pi}\cdot\frac{\mathbf{r}}{r},*$$

which is *radial and inversely proportional to the square of distance.*

* Here $\mathbf{r}/r$ is a *radial unit vector*, directed from $d\tau$ to the place at which the value ϕ holds.

Multiplying by ρ we see from the above that an element $d\tau$ behaves as a **source**, or **sink**, of productivity proportional to $\frac{d\rho}{dt}d\tau$, acting isotropically, *i.e.* emitting fluid isotropically during its dilatation, and absorbing fluid during its compression.

It is scarcely necessary to point out that such a splitting of the expression (109) into elementary summands is wholly artificial and that the corresponding 'inverse square law' is not at all characteristic of fluid motion. *Any* irrotational vector-field might be as well represented in exactly the same way.

It has already been remarked that the velocity-potential may be either a single-valued or a many-valued function of position in space.

If the fluid, in irrotational motion, occupies a *simply connected* or *acyclic* region τ, then ϕ is certainly a *single-valued* function. By the definition of the velocity-potential,

$$\phi = \int \mathbf{v}\, d\mathbf{s},$$

where the integral is to be taken along a certain line s joining the conventionally chosen 'initial' point with the point where the value of ϕ under consideration holds. Now, in the case considered it is always possible to draw through any *closed* line (s), situated wholly in the region τ, a continuous surface σ, which itself also does not pass outside the boundary of this region; consequently, by Stokes' Theorem,

$$I = \int_{(s)} \mathbf{v}\, d\mathbf{s} = 2\int \mathbf{w}\mathbf{n}\, d\sigma = 0.$$

For, at every point of the surface σ, $\mathbf{w} = \frac{1}{2}\operatorname{curl}\mathbf{v} = 0$, by hypothesis.

Let O be the 'initial' point, *i.e.* the point from which we start with the value $\phi = 0$, and let P be any other point of the region τ. Any pair of paths OAP, OBP in τ, leading from O to P may be composed to form a closed curve or a circuit $OAPBO$, which also does not pass outside the boundary of the region τ (Fig. 49). Hence

$$0 = I_{OAPBO} = I_{OAP} + I_{PBO};$$

but $I_{PBO} = -I_{OBP}$; consequently

$$I_{OAP} = I_{OBP},$$

so that independently of the path chosen we arrive at the point P with the same value of ϕ, provided only that none of the paths pass outside the boundary of the region. In other words, the

velocity-potential is, in the case of an acyclic region, a single-valued function. Q.E.D.

On the other hand, if the irrotationally moving fluid occupies a **cyclic** region, as for instance the core of an anchor-ring (doubly-connected region), the above surface σ may be drawn only through

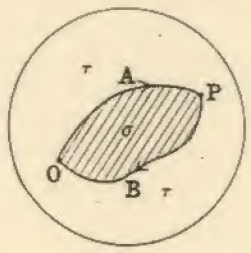

FIG. 49.

some but not through every possible closed path s contained in the region, so that in this case we shall have generally

$$\int_{(s)} \nabla\, d\mathbf{s} \neq 0,$$

i.e. a circulation generally differing from zero, and hence also, generally,

$$I_{OAP} \neq I_{OBP}.$$

Thus, the velocity-potential ϕ will for such a region be generally a *many-valued* or '**polycyc ic**' function.

An $(n+1)$-ply connected region contains n *essentially different*, or *mutually independent*, **cycles**, *i.e.* closed paths which cannot by continuous deformation without leaving the region be reduced either to a point or into each other. Call these cycles $s_1, s_2, \dots s_n$. Through none of these is it possible to draw a surface σ of the above description. Hence, the integral $\int_{(s)} \nabla\, d\mathbf{s}$ or the circulation, notwithstanding the irrotationality of motion, *may be different from zero* for each of these cycles. Let the *circulation* round $s_1, s_2, \dots s_n$ be denoted by $I_1, I_2, \dots I_n$, respectively, *i.e.*

$$\int_{(s_i)} \nabla\, d\mathbf{s} = I_i; \quad i = 1, 2, \dots n,$$

and let us again start from the initial point O with the value $\phi = 0$; by following a *certain* path OAP we shall arrive at the point P with some determinate value of the potential, $\phi = \phi_A$, where the index $_A$

indicates the path taken. Now, if we had chosen, instead of OAP, any other path OBP, we should arrive at P with a value of potential in general different from ϕ_A, namely with the value

$$\phi = \phi_A + \sum_{i=1}^{i=n} m_i I_i, \tag{110}$$

where $m_1, m_2, \dots m_n$ are integers, positive or negative, indicating the number of, positive or negative, cycles $s_1, s_2, \dots s_n$ (respectively), into which the entire circuit $OBPAO$ may be decomposed. If this circuit can be contracted to a point, by continuous deformation, without passing out of the region in consideration, then $m_1, m_2, \dots m_n$ all vanish, and $\phi = \phi_B = \phi_A$. But if $OBPAO$ cannot be contracted in this way but is reducible, for instance, to a *single* cycle s_1 taken in the *positive* sense and to the cycle s_2 repeated *three* times in the *negative* sense, then

$$m_1 = 1, \quad m_2 = -3, \quad m_3 = m_4 = \dots = m_n = 0,$$

so that

$$\phi = \phi_A + I_1 - 3I_2.$$

In the general case $m_1, m_2, \dots m_n$ may be *any mutually independent integers.*

To close this rapid survey of the properties of irrotational motion we shall further observe here that an *incompressible* fluid (shortly called a *liquid*) filling the *acyclic* interior of a *rigid shell cannot have any irrotational motion relative to the shell.*

For, in absence of sources and sinks, that is to say, if the equation

$$\nabla^2\phi = 0$$

is satisfied throughout the liquid, without any singularities, we have, by **Green's Theorem** (see 'Appendix'),

$$\int v^2 d\tau = \int (\nabla\phi)^2 d\tau = -\int \phi \mathbf{v}\mathbf{n}\, d\sigma - \int \phi \nabla^2\phi\, d\tau$$
$$= -\int \phi\, \mathbf{v}\mathbf{n}\, d\sigma,$$

the unit vector $\mathbf{n}$ denoting the normal to the shell σ drawn inwards. But, by the assumption, the shell is rigid, *i.e.* $\mathbf{v}\mathbf{n} = 0$; hence

$$\int v^2 d\tau = 0,$$

and $v = 0$ everywhere. Q.E.D.

Thus an (incompressible) liquid filling the interior of such a shell can only be in motion if it contains one or more vortices.

But in a cyclic region, such as the interior of a rigid ring-shaped tube, the liquid may move irrotationally (assuming always the absence of sources and sinks). For in this case the theorem of Green, as stated above, no longer holds, since ϕ is in general many-valued; and certain supplementary terms are required, viz. integrals of $\phi \mathbf{v}\mathbf{n}$ extending over both sides of each barrier or '*diaphragm*' necessary, and sufficient, to convert the multiply-connected into a simply-connected region. The reader will find a fuller account of this subject in Vol. I. of Basset's *Hydrodynamics*, and in Vol. I. of Maxwell's *Treatise on Electricity and Magnetism*. For the purposes of this book the above remarks on irrotational motion will suffice.

PROPERTIES OF VORTEX MOTION.

Kinematical Properties.

Any motion of a fluid may be represented as a superposition of purely *irrotational* motion characterised by

$$\operatorname{curl} \mathbf{v} = 0\,; \quad \therefore\ \mathbf{v} = \nabla\phi, \text{ but } \operatorname{div} \mathbf{v} \neq 0,$$

and of *solenoidal* vortex-motion, for which $\operatorname{curl} \mathbf{v} \neq 0$, but

$$\operatorname{div} \mathbf{v} = 0, \text{ and consequently } \mathbf{v} = \operatorname{curl} \mathbf{B}.$$

For the first kind of motion a *scalar* potential (ϕ) exists everywhere, for the second kind such a potential exists only outside the vortices, whereas the *vector* potential ($\mathbf{B}$) exists everywhere.

Hence, in the general case, to be considered in this section, we may put

$$\mathbf{v} = \nabla\phi + \operatorname{curl} \mathbf{B} \tag{111}$$

with the supplementary condition for the auxiliary vector $\mathbf{B}$, which it is always possible to satisfy,

$$\operatorname{div} \mathbf{B} = 0. \tag{112}$$

It follows for the vortex velocity, by (111),

$$\mathbf{w} = \tfrac{1}{2} \operatorname{curl} \mathbf{v} = \tfrac{1}{2} \operatorname{curl}^2 \mathbf{B} = \tfrac{1}{2} \nabla \operatorname{div} \mathbf{B} - \tfrac{1}{2} \nabla^2 \mathbf{B}, \text{ *i.e.*, by (112),}$$

$$\nabla^2 \mathbf{B} = -2\mathbf{w}, \tag{113}$$

the solution of which is, as has already been remarked,

$$\mathbf{B} = \frac{1}{2\pi} \int \frac{1}{r} \mathbf{w}\, d\tau\,; \tag{114}$$

the integration extends through all the elements $d\tau$ of the vortices, and r denotes the distance of such an element $d\tau$ from the point

in which the vector potential $\mathbf{B}$ holds. It may be easily shown that (114) satisfies not only the differential equation (113) but also the supplementary condition (112). The vector potential $\mathbf{B}$ is thus completely determined by the vortices alone.

As to the scalar potential ϕ, we have, by (111),

$$\operatorname{div} \mathbf{v} = \operatorname{div} \nabla\phi = \nabla^2\phi,$$

since $\operatorname{div} \operatorname{curl} \mathbf{B} = 0$, identically. Hence, as before, by the equation of continuity,

$$\nabla^2\phi = -\frac{d \log \rho}{dt} \tag{106 *bis*}$$

and

$$\phi = \frac{1}{4\pi} \int \frac{1}{r} \frac{d \log \rho}{dt}\, d\tau. \tag{109 *bis*}$$

This potential and the corresponding velocity have already been interpreted, in the section on irrotational motion.

Thus it will suffice to say a few words only in regard to the second part of the velocity, viz. that associated with the existence of vortices,

$$\mathbf{v} = \operatorname{curl} \mathbf{B}.$$

If the fluid is incompressible, the *whole* velocity is reduced to this. Now, by (114),

$$\mathbf{v} = \frac{1}{2\pi} \int \operatorname{curl} \left(\frac{\mathbf{w}}{r} \right) d\tau,$$

where, on curling, $\mathbf{w}$ is to be considered a constant vector, while r is variable, being the distance of $d\tau$ from that point P, for which $\mathbf{v}$ has to be calculated. Thus, denoting by $\mathbf{r}_u$ a *unit vector* pointing from $d\tau$ towards P, it follows that

$$\operatorname{curl} \left(\frac{\mathbf{w}}{r} \right) = \mathrm{V} \nabla \left(\frac{1}{r} \mathbf{w} \right) = -\frac{1}{r^2} \mathrm{V} \mathbf{r}_u \mathbf{w} = \frac{1}{r^2} \mathrm{V} \mathbf{w} \mathbf{r}_u\,;$$

hence

$$\mathbf{v} = \frac{1}{2\pi} \int \frac{1}{r^2} \mathrm{V} \mathbf{w}\, \mathbf{r}_u\, d\tau. \tag{115}$$

Taking as the volume element $d\tau$ a portion of a vortex filament of cross section $d\sigma$ and of length ds, *i.e.* putting $d\tau = d\sigma \,.\, ds$ or

$$\mathbf{w}\, d\tau = w\, d\sigma \,.\, d\mathbf{s} = \mu\, d\mathbf{s},$$

and remembering that the moment $\mu = w\, d\sigma$ is constant along the whole vortex filament, we may write (115) in a slightly different form:

$$\mathbf{v} = -\frac{1}{2\pi} \Sigma\mu \int \frac{1}{r^2} \mathrm{V} \mathbf{r}_u\, ds, \tag{115*a*}$$

where the integral is to be extended along a vortex filament and the sum Σ is to be taken over all such filaments. Splitting this expression, artificially, we may say that each element $d\mathbf{s}$ of a vortex filament 'produces' round itself or, rather, is associated with, the elementary velocity

$$-\frac{\mu}{2\pi}\frac{1}{r^2}\mathrm{V}\mathbf{r}_u d\mathbf{s},$$

i.e. with a velocity normal to the plane $\mathbf{r}$, $d\mathbf{s}$, inversely proportional to the square of the distance from the element, directly proportional to the length and to the moment of the element and, finally, proportional to the *sine* of the angle $d\mathbf{s}$, $\mathbf{r}$.

If $\mathbf{w}$ represented the electric current (per unit area) and $\mathbf{v}$ the corresponding magnetic force, the above term would be the expression of the well-known *law of Biot-Savart*.

The above formulae are not especially characteristic of the motion of a fluid or liquid nor of the electromagnetic field, but belong generally to any solenoidal vector-field regarded as dependent on its vortices.

Finally, it may be remarked here that *outside* the vortices there exists a scalar velocity-potential which, of course, is a many-valued function, also when the *whole* region τ occupied by the fluid is acyclic. If this region contains n vortices, say n vortex rings of moments $\mu_1, \mu_2, \dots \mu_n$, then, after having excluded these vortices, we obtain a cyclic, viz. $(n+1)$-ply connected region τ' (Fig. 50):

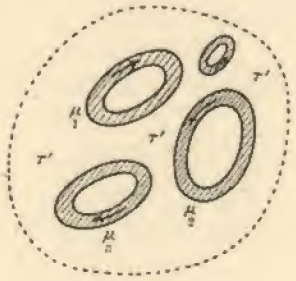

FIG. 50.

it is in this region τ' that the scalar potential of velocity exists; its form will be given by (110), where $I_1, I_2, \dots I_n$ are to be replaced by the double moments $2\mu_1, 2\mu_2, \dots 2\mu_n$, respectively.

Kinetic Energy.

Taking into account the purely solenoidal part of the motion alone or, what will amount to the same thing, supposing that the fluid is *incompressible*, ϕ will disappear from (111), and what remains is $\mathbf{v} = \operatorname{curl}\mathbf{B}$, so that the expression of the kinetic energy will be

$$L = \tfrac{1}{2}\rho\int \mathbf{v}\operatorname{curl}\mathbf{B}\,.\,d\tau.$$

But for any two vectors, as $\mathbf{v}$ and $\mathbf{B}$, we have identically (Chap. I.)

$$\mathbf{v}\operatorname{curl}\mathbf{B} - \mathbf{B}\operatorname{curl}\mathbf{v} = \operatorname{div}\mathrm{V}\mathbf{B}\mathbf{v}. \qquad (\mathrm{xx.})$$

Hence, denoting by $\mathbf{n}$ the outward normal of the bounding surface σ,

$$L = \tfrac{1}{2}\rho\int \mathbf{B}\operatorname{curl}\mathbf{v}\,.\,d\tau + \tfrac{1}{2}\rho\int \operatorname{div}\mathrm{V}\mathbf{B}\mathbf{v}\,d\tau$$

$$= \rho\int \mathbf{B}\mathbf{w}\,d\tau + \rho\int \mathbf{n}\,\mathrm{V}\mathbf{B}\mathbf{v}\,d\sigma$$

or

$$L = \rho\int \mathbf{B}\mathbf{w}\,d\tau + \rho\int \mathrm{B}\mathrm{V}\mathbf{v}\mathbf{n}\,d\sigma.$$

If the fluid has no velocity at the bounding surface σ, the surface-integral vanishes. It vanishes also if the fluid occupies the whole space, provided that all vortices, being of finite moments, are assembled in a finite portion of space. For, in this case, denoting by r the distance from any point chosen in this portion of space, and considering r to increase indefinitely,

$$B \text{ will diminish as } \frac{1}{r}, \text{ and } v \text{ as } \frac{1}{r^2},$$

hence

$$Bv \quad \text{,,} \quad \text{,,} \quad \frac{1}{r^3},$$

and since the area of the surface σ, which may be assumed to be a spherical surface of radius r, increases only as r^2, the above surface-integral will diminish as $1/r$, and will vanish for $r = \infty$.

Thus, the expression for the kinetic energy will reduce to

$$L = \rho\int \mathbf{B}\mathbf{v}\,d\tau, \qquad (116)$$

as if the seats of energy were only the rotating elements of the liquid, containing per unit volume the amount $\rho\mathbf{B}\mathbf{w}$ of this kind of energy.

Substituting the value for **B** from (114), we may write also

$$L=\frac{\rho}{2\pi}\iint\frac{\mathbf{w}\mathbf{w}'}{r}\,d\tau\,d\tau', \tag{117}$$

where r is the mutual distance of the elements $d\tau$, $d\tau'$, of which the vortex velocities are $\mathbf{w}$, $\mathbf{w}'$ (see Fig. 51).

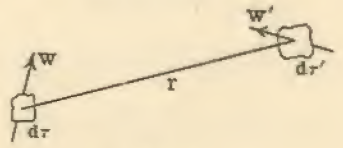

Fig. 51.

Steady Motion. Bernoulli's Theorem.

As before, let us assume that the impressed forces have a potential and that ρ depends on p only. Then the differential equation (93a) holds. Hence, if the motion be steady, *i.e.* if $\partial\mathbf{v}/\partial t=0$, we have

$$\mathrm{V}\mathbf{w}\mathbf{v}=\tfrac{1}{2}\nabla(Q-\tfrac{1}{2}v^2). \tag{118}$$

At the same time

$$\frac{\partial\rho}{\partial t}=0,$$

so that the equation of continuity (91) becomes

$$\mathrm{div}\,(\rho\mathbf{v})=0. \tag{119}$$

The lines of flow, in the present case, are immovable and coincide therefore with the stream-lines or with the paths of the fluid particles.

Multiply (118) scalarly by $\mathbf{v}$; then

$$\mathbf{v}\nabla(Q-\tfrac{1}{2}v^2)=0;$$

thus, the expression $\frac{1}{2}v^2-Q$, *i.e.*

$$\tfrac{1}{2}v^2+\Phi+\int\frac{dp}{\rho} \tag{120}$$

is constant along any line of flow; but for different lines of flow the value of this constant may be different. It is precisely the above property which is called **Bernoulli's Theorem.** It is true for any steady motion, rotational or irrotational.

In particular, for *irrotational* steady motion, *i.e.* for $\mathbf{w}=0$, we have directly from (118)

$$\nabla(Q-\tfrac{1}{2}v^2)=0,$$

whence

$$\tfrac{1}{2}v^2-Q=\text{const.},$$

the value of the 'const.' being this time one and the same *throughout the fluid.* See also (108), remembering that $\Pi=\int\frac{dp}{\rho}$.

Moreover, by (118), for *any* steady motion,

$$\mathbf{w}\nabla(Q-\tfrac{1}{2}v^2)=0,$$

so that the expression (120) *is also constant along any vortex line.* (See also '*Clebsch's Transformation*,' p. 147.)

Combining both the above results obtained from (118), we may say that

$$\tfrac{1}{2}v^2+\Phi+\Pi=\text{const.}$$

is the equation of *a surface composed of the double system of vortex lines and of lines of flow.*

In each case of *steady* motion, the whole region filled with fluid may be divided by means of such surfaces into indefinitely thin sheets.

Sir William Thomson's Theorem.

The circulation round any closed fluid filament, i.e. round any circuit composed always of the same particles, is invariable in time.

This theorem is true under the condition that the impressed forces have *a single-valued potential* and that ρ is a function of p alone, so that

$$\frac{d\mathbf{v}}{dt}=\nabla Q, \tag{93}$$

Q being a single-valued scalar function, as explained above.

To prove this theorem, let I be the circulation, *i.e.*

$$I=\int_{(s)}\mathbf{v}\,d\mathbf{s};$$

then

$$\frac{dI}{dt}=\int_{(s)}\left(\frac{d\mathbf{v}}{dt}d\mathbf{s}+\mathbf{v}\frac{d\mathbf{l}}{dt}\right),$$

where $\mathbf{l}=d\mathbf{s}$ is a line-element consisting always of the same particles

of fluid. Now, applying the formula (75) given above, in the general theory of deformation, we have

$$\mathbf{v}\frac{d\mathbf{l}}{dt} = \mathbf{v}(\mathbf{l}\nabla)\mathbf{v} = \tfrac{1}{2}(\mathbf{l}\nabla)v^2 = \tfrac{1}{2}\frac{\partial(v^2)}{\partial s}ds,$$

and, by (93),

$$\frac{d\mathbf{v}}{dt}d\mathbf{s} = \mathbf{l}\nabla Q = \frac{\partial Q}{\partial s}ds;$$

hence

$$\frac{dI}{dt} = \int_{(s)}\frac{\partial}{\partial s}(Q + \tfrac{1}{2}v^2)\,ds,$$

and since the path along which the integral is taken is closed, and Q and v^2 are *single-valued* functions of position,

$$\frac{dI}{dt} = 0, \tag{121}$$

which proves the proposition.

If then for a given instant of time $I = 0$ for every circuit, then I will remain $= 0$ for all times. In other terms: *a motion of the fluid once irrotational remains so for ever*, or as long as the assumed conditions continue to hold.

The above theorem of Thomson is closely connected with the theorems of Helmholtz, to which we shall pass in the next section. But it must be remarked that the order of presentation here followed is just the inverse of the chronological order, Thomson having written ten years after Helmholtz.*

If the condition $\rho =$ function of p only is satisfied, but if the potential of the impressed forces is a *many*-valued function or if these forces $\mathbf{F}$ have no potential at all, the circulation round an individual circuit will, in general, be *variable* in time. For, in this case,

$$\frac{dI}{dt} = \int_{(s)}\mathbf{F}\,d\mathbf{s}.$$

Still more generally we shall have, by (90),

$$\frac{dI}{dt} = \int_{(s)}\left(\mathbf{F} - \frac{1}{\rho}\nabla p\right)d\mathbf{s}, \tag{122}$$

where also the second part of the integral may differ from zero if ρ be not a function of p alone.

* H. Helmholtz, "Ueber Integrale der hydrodyn. Gleichungen, welche den Wirbelbewegungen entsprechen," *Crelle's Journal*, Vol. 55; 1858. Sir W. Thomson, "On Vortex Motion," *Trans. Roy. Soc. Edinb.*, Vol. 25; 1868.

Helmholtz's Theorems.

Let us again assume that the impressed forces have a potential and that the density is a function of the pressure alone.

Then the equation (93a),

$$\frac{\partial\mathbf{v}}{\partial t} + 2\mathrm{V}\mathbf{w}\mathbf{v} = \nabla(Q - \tfrac{1}{2}v^2),$$

holds. By curling both sides, and remembering that $\mathbf{w} = \frac{1}{2}\operatorname{curl}\mathbf{v}$,

$$\frac{\partial\mathbf{w}}{\partial t} + \operatorname{curl}\mathrm{V}\mathbf{w}\mathbf{v} = 0.$$

Now, for *any* pair of vectors $\mathbf{v}$, $\mathbf{w}$,

$$\operatorname{curl}\mathrm{V}\mathbf{w}\mathbf{v} = \mathbf{w}\operatorname{div}\mathbf{v} + (\mathbf{v}\nabla)\mathbf{w} - \mathbf{v}\operatorname{div}\mathbf{w} - (\mathbf{w}\nabla)\mathbf{v}. \tag{XXI.}$$

In the present case $\mathbf{w} = \frac{1}{2}\operatorname{curl}\mathbf{v}$, and consequently $\operatorname{div}\mathbf{w} = 0$: hence

$$0 = \frac{\partial\mathbf{w}}{\partial t} + (\mathbf{v}\nabla)\mathbf{w} + \mathbf{w}\operatorname{div}\mathbf{v} - (\mathbf{w}\nabla)\mathbf{v}$$

$$= \frac{d\mathbf{w}}{dt} + \mathbf{w}\operatorname{div}\mathbf{v} - (\mathbf{w}\nabla)\mathbf{v},$$

or, by the equation of continuity, (91),

$$\frac{d\mathbf{w}}{dt} - \frac{\mathbf{w}}{\rho}\frac{d\rho}{dt} = (\mathbf{w}\nabla)\mathbf{v},$$

which may, finally, be written

$$\frac{d}{dt}\left(\frac{\mathbf{w}}{\rho}\right) = \frac{1}{\rho}(\mathbf{w}\nabla)\mathbf{v}. \tag{123}$$

The truth of the celebrated theorems of Helmholtz may easily be seen to follow from this equation, which is the vector equivalent of Helmholtz's three scalar equations.

In fact, let $\mathbf{l}$ be, in direction and length, an element of a *vortex filament*, of infinitesimal cross section σ. Denoting by ϵ an infinitesimal scalar, we may write

$$\mathbf{l} = \epsilon\mathbf{w},$$

and, by the general formula (75),

$$\frac{d\mathbf{l}}{dt} = \epsilon(\mathbf{w}\nabla)\mathbf{v};$$

hence, by (123),

$$\frac{d}{dt}\left(\frac{\mathbf{w}}{\rho}\right) = \frac{1}{\rho\epsilon}\frac{d\mathbf{l}}{dt}.$$

Let the moment of the vortex filament be μ, *i.e.* $\mu = \sigma w$. The mass ϵ of the element, $\epsilon = \rho/\sigma$, is constant in time, and since $\epsilon = l/w$, we may write

$$\rho\epsilon = \rho l/w = c/\sigma w = c/\mu,$$

and consequently

$$\frac{d}{dt}\left(\frac{\mathbf{w}}{\rho}\right) = \frac{\mu}{c}\frac{d\mathbf{l}}{dt}. \qquad (a)$$

But $\mathbf{w}/\rho = w\mathbf{l}/l\rho = w\sigma \cdot \mathbf{l}/c = \mu\mathbf{l}/c$, identically. Hence, multiplying both sides of (a) by the constant mass c,

$$\frac{d}{dt}(\mu\mathbf{l}) = \mu\frac{d\mathbf{l}}{dt};$$

whence, finally:

$$\frac{d\mu}{dt} = 0, \qquad (124)$$

i.e.:

*The moment of a vortex filament is invariable in time.**

By (124), equation (a) may now be written

$$\frac{d}{dt}\left(\mathbf{l} - \frac{c}{\mu\rho}\mathbf{w}\right) = 0$$

or

$$\frac{d}{dt}(\mathbf{l} - \epsilon\mathbf{w}) = 0;$$

but, for a given instant of time t we had $\mathbf{l} = \epsilon\mathbf{w}$; hence, *for all times*: $\mathbf{l} = \epsilon\mathbf{w}$, that is to say: if the element $\mathbf{l}$, which consists always of the same particles, has been an element of a vortex line at one instant only, then it will be so at any other instant of time. In other words:

Every vortex line and, consequently, also every vortex tube consists always of the same particles of fluid.

Particles that do not rotate at a given instant have never been rotating and never will rotate, as long, of course, as the conditions stated above hold.

These are the theorems of Helmholtz, which express the *indestructibility* and the *uncreatability* of vortices, in a non-viscous fluid which is acted on solely by conservative forces, and of which the density is a function of the pressure alone.

* That μ is also constant *along* the whole vortex filament has been proved already, quite independently of any dynamical properties.

The second of the above (italicised) theorems has, also, been proved already, in the section on Clebsch's Transformation.

Moreover, the invariability of the moment of a vortex follows very directly from the constancy of circulation $I = \int \mathbf{v}\, d\mathbf{s}$; for, taking the circuit s, composed always of the same particles, on the surface of a given vortex tube of moment μ, and so that the circuit embraces the tube once only (in the positive sense), we have $I = 2\mu$; so that the equation (121), $\frac{dI}{dt} = 0$, assumes at once the required form

$$\frac{d\mu}{dt} = 0.$$

Again, without impairing the equation $I = 2\mu$, we may take the circuit s large at random, provided only that it does not embrace any other vortex tubes but the one in question. Hence:

If a closed fluid filament embraces at a given instant some vortex tube, it will embrace this tube for ever.

Also the *number of times* that the vortex tube is embraced by the circuit *does not change* in time; for, if it is m (positive or negative), then $I = 2m\mu$, and since both I and μ are invariable, m too is invariable.

Creation of Vortices along the Intersections of Surfaces of Constant Density and Constant Pressure.

Helmholtz's theorems do not hold if the impressed forces have no potential or if ρ is no longer a function of p alone.

Instead of (93a) we have then the more general equation,

$$\frac{\partial \mathbf{v}}{\partial t} + 2\mathbf{V}\mathbf{w}\mathbf{v} = \mathbf{F} - \frac{1}{\rho}\nabla p - \tfrac{1}{2}\nabla(v^2),$$

and, consequently, instead of (123),

$$\rho\frac{d}{dt}\left(\frac{\mathbf{w}}{\rho}\right) = (\mathbf{w}\nabla)\mathbf{v} + \tfrac{1}{2}\operatorname{curl}\mathbf{F} - \tfrac{1}{2}\operatorname{curl}\left(\frac{1}{\rho}\nabla p\right).$$

But, as we saw, when treating of Clebsch's Transformation,

$$\operatorname{curl}\left(\frac{1}{\rho}\nabla p\right) = \mathbf{V}\nabla\left(\frac{1}{\rho}\right).\nabla p = -\frac{1}{\rho^2}\mathbf{V}\nabla\rho\,.\,\nabla p.$$

Hence, as a generalisation of (123),

$$\rho\frac{d}{dt}\left(\frac{\mathbf{w}}{\rho}\right) - (\mathbf{w}\nabla)\mathbf{v} = \tfrac{1}{2}\operatorname{curl}\mathbf{F} + \frac{1}{2\rho^2}\mathbf{V}\nabla\rho\,.\,\nabla p. \qquad (125)$$

If we were to suppose again that ρ is a function of p alone, the last term would vanish; hence, when a given particle of the fluid *does not rotate* at the instant t_0, then *for this instant*

$$\frac{d\mathbf{w}}{dt} = \tfrac{1}{2}\operatorname{curl}\mathbf{F};$$

thus, the particle *will begin to rotate*, namely round an axis which initially (*i.e.* for $t = t_0$) will coincide in direction with the vortex of the force $\mathbf{F}$, *i.e.* with the vector $\operatorname{curl}\mathbf{F}$. Also the moment μ of fluid vortices already existing will generally vary with time.

From the last equation we see also that if the motion is to be *irrotational*, *i.e.* if we require that $w = 0$ *permanently*, we must have $\operatorname{curl}\mathbf{F} = 0$.

But let us suppose that *the forces* $\mathbf{F}$ *have a potential*, and that ρ *is not a function of* p *alone*. It may, for instance, depend also on the temperature which may be different for different particles and at various instants of time. In this case the first term on the right side of (125) vanishes, but the second remains, so that

$$\rho\frac{d}{dt}\left(\frac{\mathbf{w}}{\rho}\right) - (\mathbf{w}\nabla)\mathbf{v} = \frac{1}{2\rho^2}\mathrm{V}\nabla\rho\,.\,\nabla p. \qquad (126)$$

If $\mathbf{n}$, $\mathbf{n}'$ are unit vectors normal to the surfaces $\rho = const.$, $p = const.$ respectively, drawn in the directions of *increasing* density and of increasing pressure, then (126) may also be written

$$\rho\frac{d}{dt}\left(\frac{\mathbf{w}}{\rho}\right) - (\mathbf{w}\nabla)\mathbf{v} = \frac{1}{2\rho^2}\frac{\partial\rho}{\partial n}\frac{\partial p}{\partial n'}\mathrm{V}\mathbf{n}\mathbf{n}'. \qquad (126a)$$

The vector on the right side, and consequently also the vector sum on the left side, is tangent to the line of intersection of the surfaces $\rho = \text{const.}$ and $p = \text{const.}$

Considering again an element of a vortex filament $\mathbf{l} = \epsilon\mathbf{w}$, we may transform the left side of (126) into

$$\frac{1}{c}\frac{d(\mu\mathbf{l})}{dt} - \frac{\mu}{c}\frac{d\mathbf{l}}{dt} \quad \text{or} \quad \frac{\mathbf{l}}{c}\frac{d\mu}{dt},$$

c being the mass of the element of length l and of cross section σ, and $\mu = \sigma\kappa$ being the moment of the vortex filament at the instant t. Hence

$$\mathbf{l}\frac{d\mu}{dt} = \frac{c}{2\rho^3}\frac{\partial\rho}{\partial n}\frac{\partial p}{\partial n'}\mathrm{V}\mathbf{n}\mathbf{n}$$

or

$$l\frac{d\mu}{dt} = \frac{c}{2\rho^3}\frac{\partial\rho}{\partial n}\frac{\partial p}{\partial n'}\sin(\mathbf{n},\mathbf{n}'). \qquad (127)$$

Thus, *the moment of the vortex filament will, in general, change with the time*, unless the angle $\theta = \mathbf{n}, \mathbf{n}'$ vanishes (or is $= \pi$) or unless the surfaces $\rho = \text{const.}$, $p = \text{const.}$ coincide with one another. But if they intersect mutually, at the given place, the moment μ *increases* or *decreases* with t, according as $\sin\theta > 0$ or $\sin\theta < 0$.

Thus, a rotating particle may in time lose its vortex motion.

On the other hand, if at a given instant t_0, for some particle, or for the whole fluid, $w = 0$, then, by (126),

$$\frac{d\mathbf{w}}{dt} = \frac{1}{2\rho^2}\frac{\partial\rho}{\partial n}\frac{\partial p}{\partial n'}\mathrm{V}\mathbf{n}\mathbf{n}', \text{ for } t = t_0, \qquad (128)$$

that is to say:

In a fluid devoid, at the instant t_0, or till the instant t_0, of every vortex motion, *vortex lines would be created, coinciding initially with the intersections of the surfaces of constant density and of constant pressure.**

If such surfaces, intersecting mutually during a certain time T, should then once and for ever cease to intersect one another, the vortices created during that time would afterwards keep their moments constant. The period T might be brief, yet the moments of the vortices which have been formed in this short time might be considerable when the gradients of pressure and of density are correspondingly large.

The Wave of Acceleration; Hugoniot's Theorem.

The *surfaces of discontinuity*, for which the identical and the kinematical conditions have already been established (pp. 122, 131), must also fulfil certain conditions following from the *dynamical* properties of a non-rigid body, and, in particular, of a fluid.

We shall limit ourselves here to the consideration of a surface of discontinuity of the *second order* (in $\mathbf{r}$, *i.e.* of the *first* order in $\mathbf{v} = d\mathbf{r}/dt$) or of the so-called **wave of acceleration**, which was generally investigated by *Hugoniot*.†

Let ρ be a function of p only; then

$$\frac{1}{\rho}\nabla p = \frac{dp}{d\rho}\frac{1}{\rho}\nabla\rho = \frac{dp}{d\rho}\nabla\log\rho;$$

* Cf. the author's paper, 'Ueber Entstehung von Wirbelbewegungen in einer reibungslosen Flüssigkeit,' *Bulletin Acad. Sc. Cracow*, 1896.

† *Comptes rendus*, Vol. 101, p. 1119; Paris, 1885.

hence the equation of motion:

$$\frac{d^2\mathbf{r}}{dt^2}=\frac{d\mathbf{v}}{dt}=\mathbf{F}-\frac{dp}{d\rho}\nabla\log\rho. \tag{129}$$

Let, again,

$$f=f(a,\ t)=0$$

be the equation of the surface of discontinuity σ, and $\mathbf{n}$ its normal. Let the impressed forces $\mathbf{F}$, the velocity $\mathbf{v}$, the pressure and density, as well as $dp/d\rho$ be continuous, whereas the *acceleration* (and hence also $\nabla\log\rho$) is *discontinuous* at the surface σ.

Then, writing (129) for the two sides of σ and subtracting, we shall have

$$\left[\frac{d\mathbf{v}}{dt}\right]=-\frac{dp}{d\rho}[\nabla\log\rho]. \tag{a}$$

Since $[\mathbf{v}]=0$, we have, in exactly the same way as in eq. (72) for $\mathbf{D}$,

$$[\operatorname{div}\mathbf{v}]=\mathbf{mn}=\mathbf{m}\nabla f:\frac{\partial f}{\partial n}, \tag{b}$$

where $\mathbf{m}$ is the vector which characterises the discontinuity.

On the other hand, by (70), since $[\log\rho]=0$,

$$[\nabla\log\rho]=\lambda\mathbf{n}, \tag{c}$$

λ being a scalar which we shall eliminate immediately. Again, the kinematical condition of compatibility will be, as in (82),

$$\left[\frac{d\log\rho}{dt}\right]=-\lambda\mathfrak{v}, \tag{d}$$

where $\mathfrak{v}$ is the velocity of propagation of the wave, the value of which is to be found.

But, by the equation of continuity, (91), $d\log\rho/dt=-div\,\mathbf{v}$, and consequently

$$\left[\frac{d\log\rho}{dt}\right]=-[\operatorname{div}\mathbf{v}],$$

so that, by (*b*) and (*d*),

$$\lambda=\frac{1}{\mathfrak{v}}\mathbf{mn};$$

hence, by (*c*),

$$[\nabla\log\rho]=\frac{1}{\mathfrak{v}}(\mathbf{mn})\mathbf{n}. \tag{130}$$

This is the vector equivalent of the three scalar formulae for the jump of the components of slope of density, obtained by Hadamard by a considerably longer process.*

* See Hadamard, *loc. cit.*, or also Appell's *Mécanique*, Vol. III. p. 312.

On the other hand we have, as in (82),

$$\left[\frac{d\mathbf{v}}{dt}\right]=-\mathfrak{v}\mathbf{m};$$

hence, introducing this value and also (130) into the dynamical equation (*a*),

$$\mathfrak{v}^2\mathbf{m}=\frac{dp}{d\rho}(\mathbf{mn})\mathbf{n}. \tag{131}$$

There are only two possible ways of satisfying this equation:

1) the vector $\mathbf{m}$ is normal to the surface σ, *i.e.* $\mathbf{m}=m\mathbf{n}$, and at the same time

$$\mathfrak{v}=\sqrt{\frac{dp}{d\rho}},$$

or 2) $\mathbf{m}$ is tangent to σ, or $\mathbf{mn}=0$, and at the same time

$$\mathfrak{v}=0.$$

In the first case the discontinuity is called **longitudinal**, in the second, **transversal**. Thus we have the **Theorem of Hugoniot**:

In a compressible, non-viscous fluid there are possible only two kinds of discontinuities of the second order: longitudinal, which are propagated with the velocity

$$\mathfrak{v}=\sqrt{\frac{dp}{d\rho}}, \tag{132}$$

and transversal, which are not propagated at all, i.e. which affect always the same particles.

The first are *waves* in the proper sense of the word, while the second are *stationary* discontinuities.

(132) is the celebrated **formula of Laplace** for the velocity of sound in gases.

It may further be observed that for longitudinal waves, by (*b*),

$$[\operatorname{div}\mathbf{v}]\neq 0,$$

namely

$$[\operatorname{div}\mathbf{v}]=\mathbf{mn}=m\mathbf{n}^2=m,$$

whereas for transversal discontinuities

$$[\operatorname{div}\mathbf{v}]=0.$$

As in (73), $[\operatorname{curl}\mathbf{v}]=\mathrm{V}\mathbf{nm}$. Thus, in the first case we shall have $[\operatorname{curl}\mathbf{v}]=0$, and in the second $[\operatorname{curl}\mathbf{v}]\neq 0$.

Hence, to sum up these last conclusions, we see that for longitudinal waves $\operatorname{div}\mathbf{v}$ jumps while the vortex velocity $\mathbf{w}$ is continuous, whereas for transversal discontinuities, attached always to the same fluid particles, the opposite is the case.

PROBLEMS AND EXERCISES.

THE arrangement of these 'Problems and Exercises' is parallel to the sequence of ideas in the above six chapters. This remark may aid the reader. Besides, special hints will be given, in small print, where needed. The reader should attempt to solve each problem without any artificial splitting of the vectors involved into Cartesians, unless the contrary is expressly stated, as when a problem or exercise is to consist in the translation of a vector formula into Cartesian language. The decomposition of a vector into components taken along or perpendicularly to other vectors involved in a problem is, of course, not 'artificial'; nor has, for instance, the splitting of a vector into components parallel to the principal axes of a linear vector operator anything artificial about it. The reader is recommended to verify his results, in doubtful cases, by Cartesian developments, at least in the beginning of his vectorial career. But he will soon find that he can trust his vectorial calculation sufficiently, so as not to be obliged to recur to any translation whatever. He will 'think vectorially' as we begin to 'think in a foreign language' after we have learned it sufficiently,—the only difference being that the vector language can be learned (by people knowing the Cartesian or any mathematical language at all) much more easily than an ordinary, spoken, foreign language, such as French or German.

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1. Prove that the diagonals of a parallelogram bisect each other.

2. *Vice versa*, the diagonals of a given quadrilateral bisecting one another, prove that the figure is a parallelogram.

3. Prove that the eight vectors

$$\pm\mathbf{A}\pm\mathbf{B}\pm\mathbf{C},$$

i.e. $\mathbf{A}+\mathbf{B}+\mathbf{C}$, $\mathbf{A}+\mathbf{B}-\mathbf{C}$, etc., when drawn through a common point terminate at the vertices of a parallelopiped. (Joly, *Quaternions.*)

4. Prove that if in any quadrilateral the mid-points of the sides be joined in order, the figure formed is a parallelogram. (Henrici and Turner, *Vectors and Rotors.*)

5. Prove that the four diagonals of a parallelopiped meet and bisect each other. (*Ibidem.*)

6. Prove the famous **Theorem of Desargues**: If ABC and $A'B'C'$ are two triangles (coplanar or not) such that the straight lines AA', BB', CC' meet in a point O, the points of intersection of AB and $A'B'$, of BC and $B'C'$, of CA and $C'A'$ lie in a straight line.

Take O as the origin of your vectors. Call $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{a}'$, etc., the vectors OA, OB, etc. Then $\mathbf{a}'=\alpha\mathbf{a}$, $\mathbf{b}'=\beta\mathbf{b}$, $\mathbf{c}'=\gamma\mathbf{c}$, where α, β, γ are scalars. Let $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$ be the vectors drawn from O to the points of intersection of BC and $B'C'$, etc.; show that then

$$\mathbf{l}=\mathbf{c}+(\mathbf{b}-\mathbf{c})\frac{\beta(1-\gamma)}{\beta-\gamma}, \text{ etc.},$$

whence, and so on.

The proof of this and of all the above theorems involves only the addition, and subtraction, of vectors.

7. Similarly, *mutatis mutandis*, show that the converse of the last theorem is equally true.

8. Prove that if two tetrahedra $ABCD$ and $A'B'C'D'$ are perspective from a point, they are perspective from a plane.

This means that if AA', BB', etc., meet in a point O, the six pairs of homologous edges intersect in coplanar points and the four pairs of homologous faces intersect in coplanar lines (cf. Veblen and Young's *Projective Geometry*, Vol. I. p. 43; Boston, etc., 1910). Take again O as the origin of your vectors, etc. Of course, your vectorial method will be far less elegant than that used by projective geometry to prove this and the above two projective theorems; but these exercises will still be found useful and interesting.

9. Describe, in words, the different meanings of the products

$$\mathbf{A}(\mathbf{BC}),\quad (\mathbf{AB})\mathbf{C},\quad (\mathbf{CA})\mathbf{B},$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are three arbitrary vectors. Also, write down their Cartesian equivalents.

10. If $\mathbf{A}$ and $\mathbf{B}$ be any two vectors, what is the meaning of the vector expression

$$\mathbf{A} - \frac{(\mathbf{AB})\mathbf{B}}{B^2}?$$

11. $\mathbf{C}$ being the vector sum of $\mathbf{A}$ and $\mathbf{B}$, interpret geometrically the meaning of the identity

$$C^2 = A^2 + B^2 + 2\mathbf{AB}.$$

12. Put

$$(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B}) = A^2 - B^2$$

into the form of a geometrical theorem.

13. If $A = B$, then the above equation degenerates into

$$(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B}) = 0.$$

What does it mean geometrically?

14. By the distributive property of scalar multiplication,

$$(\mathbf{A}+\mathbf{B})\mathrm{V}\mathbf{CD} = \mathbf{A}\mathrm{V}\mathbf{CD} + \mathbf{B}\mathrm{V}\mathbf{CD}.$$

Interpret it geometrically (stereometrically).

Make $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ coinitial.

15. If $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ be the system of vectors defined in Chap. I., what are the values of $\mathbf{i}\mathrm{V}\mathbf{jk}$, $\mathbf{j}\mathrm{V}\mathbf{ki}$, $\mathbf{k}\mathrm{V}\mathbf{ij}$, $\mathbf{i}\mathrm{V}\mathbf{kj}$, etc.?

16. Prove the distributive property of vector multiplication (Theor. IV. Chap. I.) by geometrical construction.

17. What is the geometrical meaning of the equality

$$\mathrm{V}(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B}) = 2\mathrm{V}\mathbf{BA}?$$

18. Prove vectorially the fundamental formulae of plane trigonometry,

$$\sin(\alpha \pm \beta) = \sin\alpha \,.\, \cos\beta \pm \sin\beta \,.\, \cos\alpha,$$
$$\cos(\alpha \pm \beta) = \cos\alpha \,.\, \cos\beta \mp \sin\alpha \,.\, \sin\beta.$$

19. Any vector $\mathbf{r}$ may be represented in the form

$$\mathbf{r} = s_1\mathbf{a} + s_2\mathbf{b} + s_3\mathbf{c},$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is an arbitrary set of three non-coplanar vectors, say, unit vectors, and where s_1, s_2, s_3 are appropriate scalars. Determine them. Show that any $\mathbf{r}$ may be written

$$\mathbf{r} = \frac{1}{\mathbf{a}\mathrm{V}\mathbf{bc}}\{(\mathbf{r}\mathrm{V}\mathbf{bc})\mathbf{a} + (\mathbf{r}\mathrm{V}\mathbf{ca})\mathbf{b} + (\mathbf{r}\mathrm{V}\mathbf{ab})\mathbf{c}\}.$$

(Heaviside, *Electromagnetic Theory.*)

20. Prove the formula (4) of Chap. I. without splitting the factors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ into Cartesians.

Take first the simpler case of $\mathbf{A} \| \mathbf{B}$, *i.e.* $\mathbf{b}$, $\mathbf{c}$ being unit vectors, show that

$$\mathrm{V}\mathbf{b}\mathrm{V}\mathbf{bc} = \mathbf{b}(\mathbf{bc}) - \mathbf{c},$$

hence also

$$\mathrm{V}\mathbf{c}\mathrm{V}\mathbf{bc} = -\mathrm{V}\mathbf{c}\mathrm{V}\mathbf{cb} = \mathbf{b} - \mathbf{c}(\mathbf{bc}).$$

Then decomposing $\mathbf{A}$ along $\mathbf{b}$, $\mathbf{c}$ (such a decomposition having nothing 'artificial' about it), prove the general formula. Notice that the component of $\mathbf{A}$ normal to the plane $\mathbf{B}$, $\mathbf{C}$ is inoperative.

21. What conclusions may be drawn from the equation

$$\mathrm{V}\mathbf{A}\mathrm{V}\mathbf{BC} = 0$$

with regard to the three, otherwise unknown, factors?

Discuss fully all possibilities.

22. Three particles, 1, 2, 3, are moving relatively to a fixed point O with any (variable) velocities $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$. Find the time-rate of change, p, of the area of the triangle 123. If $\mathbf{p}$ be the vector normal to the plane 123 and having the above p for its tensor, find the rate of change of $\mathbf{p}$ in terms of the velocities and accelerations of the three particles.

23. Similarly, $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ being the velocities of four particles, find the rate of change, q, of the volume of the tetrahedron 1234, and the rate of change of q.

24. Prove that the **hodograph** * of the parabolic motion,

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}_0 t + \tfrac{1}{2}\mathbf{a}t^2$$

(where $\mathbf{r}_0$, $\mathbf{v}_0$, $\mathbf{a}$ are constant vectors, whose meaning is obvious), is a straight line described uniformly.

25. Investigate the properties of the so-called elliptic harmonic motion,

$$\mathbf{r} = \mathbf{A}\sin nt + \mathbf{B}\cos nt,$$

where $\mathbf{A}$, $\mathbf{B}$ is a given pair of constant, and generally oblique vectors, and n a constant scalar. Find the hodograph of this motion. (Clifford, *Elements of Dynamic.*)

26. Write down the formulae for the acceleration, for the last two cases of motion, exhibiting its tangential and normal components.

* If a straight line Op be drawn through a fixed point O, representing at every instant the velocity $\mathbf{v}$ of a moving point P, the curve described by p is called the *hodograph* of the motion of P.

27. Let s be any circuit (having no knots in it) and $\mathbf{r}$ the vector drawn from any fixed point O to any point of s. Investigate the properties of the integral

$$\mathbf{I} = c\int \mathrm{V}\mathbf{r}\,d\mathbf{s},$$

taken round s, where c is an arbitrary scalar constant. Show that $\mathbf{I}$ is an *invariant* of the circuit with respect to the choice of O, *i.e.* that, taking O' instead of O, the corresponding $\mathbf{I}'$ is equal to $\mathbf{I}$. Join any two points of s by a curve, thus decomposing the circuit s into two others, s_1, s_2; show that then

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2, \quad i.e. \quad \mathbf{I}(s) = \mathbf{I}(s_1) + \mathbf{I}(s_2).$$

27*a*. In particular, if s be a square, show that $\mathbf{I} = 2c\mathbf{n}$, where $\mathbf{n}$ is a unit vector normal to the plane of the square. (Fix the right sense of $\mathbf{n}$.) Hence, defining the 'area' of a square to be unity (so that $c = \frac{1}{2}$), and using the above additive property, interpret the geometrical meaning of $\frac{1}{2}\int \mathrm{V}\mathbf{r}\,d\mathbf{s}$ for any, and especially for plane, circuits s.*

In the case of the square take your O in one of its corners.

28. If $\mathbf{r}$ be the vector drawn from a fixed point O to any point of space, show (without splitting) that $\mathrm{curl}\,\mathbf{r} = 0$.

Use the form $\mathrm{curl} = \mathrm{V}\nabla$, show that ∇r is a radial (unit) vector, and so on.

29. The meaning of $\mathbf{r}$ being as above, show that, for any circuit s,

$$\int \mathbf{r}\,d\mathbf{s} = 0.$$

30. Using the definition of *curl* and of *div* by means of the line and surface integrals, prove their distributive property.

31. Show that the volume of a portion of space enclosed by a surface σ is

$$\tau = \tfrac{1}{3}\int \mathbf{r}\mathbf{n}\,d\sigma,$$

* Extending the usual definition of *moment* of a finite straight line (see *e.g.* Clifford's *Dynamics*, Part I. p. 92), we might call the integral $\int \mathrm{V}\mathbf{r}\,d\mathbf{s}$ the *moment of the circuit s* (about the point O, whose specification, however, is in our case irrelevant). As a supplement to the above exercise consider the properties of the moment, $\int \mathrm{V}\mathbf{r}\,d\mathbf{s}$, of an *open* curve about a given point O, and investigate its dependence on the choice of O.

$\mathbf{n}$ being the outward unit normal and $\mathbf{r}$ a vector drawn from a fixed point O to any point of σ. Verify this result by putting, in Theor. VI., $\mathbf{R} = \mathbf{r}$. Show directly that $\int \mathbf{r}\mathbf{n}\,d\sigma$ is an invariant with respect to the choice of O (inside or outside σ).

Compare, by the way, the properties of this integral with that involved in 27 and 27*a*. As to the immediate evaluation of $\mathrm{div}\,\mathbf{r}$, write it $\nabla\mathbf{r}$, and so on. To prove the invariance of the above integral, show first that $\int \mathbf{n}\,d\sigma = 0$.

32. The equation of a surface being $f = 0$, where f is a given scalar function of position in space, show that

$$\frac{\mathrm{V}\nabla f\,\mathrm{V}\mathbf{R}\nabla f}{(\nabla f)^2} \quad \text{and} \quad \frac{\nabla f(\mathbf{R}\nabla f)}{(\nabla f)^2}$$

are the parts of $\mathbf{R}$ tangential and normal (respectively) to the surface, both in size and in direction.

33. Find the value of

$$\mathrm{div}\,\{\phi(r)\mathbf{r}\},$$

using only the definition of *div* by means of the surface integral. (Here $\mathbf{r}$ has the same meaning as in 28, etc., and ϕ is any given function of its tensor r.)

34. Express the slope of a scalar function of position ϕ in terms of *orthogonal curvilinear coordinates* u, v, w. That is to say, $u = \mathrm{const.}$, $v = \mathrm{const.}$, $w = \mathrm{const.}$ being a threefold orthogonal system of surfaces in space, show that

$$\nabla\phi = \mathbf{U}\frac{\partial\phi}{\partial u} + \mathbf{V}\frac{\partial\phi}{\partial v} + \mathbf{W}\frac{\partial\phi}{\partial w},$$

where the vectors $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$ are the slopes of u, v, w respectively, *i.e.*

$$\mathbf{U} = \nabla u, \quad \mathbf{V} = \nabla v, \quad \mathbf{W} = \nabla w.$$

Do not employ the expression of ∇ in terms of $\partial/\partial x$, etc. Use only (x.), Chap. I., as the definition of slope. If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, be the units of $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$ respectively, and ds_1, ds_2, ds_3 the elements of the lines of intersection of the surfaces $v = \mathrm{const.}$, $w = \mathrm{const.}$, etc., measured in the direction of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, then $\nabla\phi = \mathbf{a}\,\partial\phi/\partial s_1 + \ldots$, and so on. Notice that the conditions of orthogonality are expressed simply by

$$\mathbf{VW} = \mathbf{WU} = \mathbf{UV} = 0, \quad i.e. \quad \nabla v \,.\, \nabla w = 0, \text{ etc.}$$

35. U, V, W being the tensors of the above $\mathbf{U}$, etc., show that

$$\mathrm{div}\,\mathbf{R} = UVW\left\{\frac{\partial}{\partial u}\left(\frac{R_1}{VW}\right) + \frac{\partial}{\partial v}\left(\frac{R_2}{WU}\right) + \frac{\partial}{\partial w}\left(\frac{R_3}{UV}\right)\right\},$$

where $R_1 = \mathbf{Ra}$, etc., are the components of the vector $\mathbf{R}$ along the lines of intersection of the surfaces $v = \text{const.}$, $w = \text{const.}$, etc.

Using the definition of *div* by means of the surface integral, consider the element of volume bounded by the three pairs of surfaces u and $u+du$, v and $v+dv$, w and $w+dw$; remember that $ds_1 = du/U$, and so on.

36. Show that the Laplacian in orthogonal curvilinear coordinates is

$$\nabla^2 = UVW\left\{\frac{\partial}{\partial u}\left(\frac{U}{VW}\frac{\partial}{\partial u}\right)+\frac{\partial}{\partial v}\left(\frac{V}{WU}\frac{\partial}{\partial v}\right)+\frac{\partial}{\partial w}\left(\frac{W}{UV}\frac{\partial}{\partial w}\right)\right\}.$$

Use the form $\nabla^2\phi = \operatorname{div}\nabla\phi$ and apply the above result.

37. Retaining the above notation and assuming the order of u, v, w, so that $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ should be a *right-handed* system, show that the curvilinear components of curl $\mathbf{R}$ are

$$VW\left\{\frac{\partial}{\partial v}\left(\frac{R_3}{W}\right)-\frac{\partial}{\partial w}\left(\frac{R_2}{V}\right)\right\}, \text{ etc.,}$$

or that, in determinant form,

$$\operatorname{curl}\mathbf{R} = UVW\begin{vmatrix} \frac{\mathbf{a}}{U} & \frac{\mathbf{b}}{V} & \frac{\mathbf{c}}{W} \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ \frac{R_1}{U} & \frac{R_2}{V} & \frac{R_3}{W} \end{vmatrix}.$$

Use the definition of *curl* by means of the line integral; take an infinitesimal rectangle v, $v+dv$, w, $w+dw$ on a u-surface, and so on.

38. Taking for the above u-, v-, w-surfaces confocal ellipsoids, one-sheeted and two-sheeted hyperboloids, write down *div*, *curl* and *Laplacian* in *elliptic coordinates*.

In this case,

$$U^2 = \frac{f(u)}{(u-v)(u-w)},$$

with two similar expressions for V^2 and W^2, where

$$f(u) = 4(A^2+u)(B^2+u)(C^2+u),$$

A, B, C being constants for the whole system; $A^2 > B^2 > C^2$; $2A$, $2B$, $2C$ are the principal axes of the ellipsoid $u = 0$. If x, y, z are the ordinary, rectilinear coordinates measured along the principal axes of the whole

system of confocal quadric surfaces, $u > v > w$ are the roots of the equation

$$\frac{x^2}{A^2+\lambda}+\frac{y^2}{B^2+\lambda}+\frac{z^2}{C^2+\lambda} = 1,$$

$$\infty > u > -C^2 > v > -B^2 > w > A^2,$$

so that $f(u)$, $f(w)$ are positive, while $f(v)$ is negative. Remembering all of this, use 35, 37 and 36, respectively. Here it will be enough to write the Laplacian, leaving the *curl* and *div* wholly to the reader. Introducing the elliptic integrals

$$\alpha = \int\frac{du}{\sqrt{f(u)}}, \quad \beta = \int\frac{dv}{\sqrt{-f(v)}}, \quad \gamma = \int\frac{dw}{\sqrt{f(w)}},$$

the Laplacian will be

$$\nabla^2 = N\left\{(v-w)\frac{\partial^2}{\partial\alpha^2}+(u-w)\frac{\partial^2}{\partial\beta^2}+(u-v)\frac{\partial^2}{\partial\gamma^2}\right\},$$

where $N^{-1} = (v-w)(u-w)(u-v)$.

39. Write *div*, *curl*, ∇^2 in *polar coordinates* $u = \theta$, $v = \epsilon$, $w = r$.

In this case, θ, ϵ being the latitude and longitude respectively, $ds_1 = r\,d\theta$, $ds_2 = r\sin\theta\,.\,d\epsilon$, $ds_3 = dr$, so that $U = 1/r$, $V = 1/r\sin\theta$, $W = 1$, and so on.

40. Write down *div*, *curl*, ∇^2 in *cylindrical coordinates* $u = z$, $v = \rho$, $w = \epsilon$.

Here, ρ being the distance from, and z being measured along, the axis, $U = 1$, $V = 1$, $W = 1/\rho$. Consider the ~~peculiarly~~ important special case of *axial symmetry*, *i.e.* $\partial/\partial\epsilon = 0$.

41. Verify, by Cartesian expansion, the formula (26) of Chap. I. Show, without splitting, that the divergence of its right-hand side vanishes identically.

42. Verify, by Cartesian expansion, the formula (33) of Chap. I. Write down curl^3, curl^4, etc., for a solenoidal vector.

43. Show that d'Alembert's Principle, (1), Chap. II., is invariant with respect to the 'Newtonian transformation' $\mathbf{r}_i' = \mathbf{r}_i - \mathbf{a}t$, where $\mathbf{a}$ is an arbitrary constant vector.

Assume, of course, that the constraints of the system imply only the positions, and velocities, of its constituent parts relative to one another.

44. Formula (1), Chap. II., being the expression of d'Alembert's Principle relative, say, to the 'fixed-star' system of reference, write this principle using the earth as a framework of reference. Compare it with (1) and interpret the supplementary terms.

45. Integrate the equation of motion of a particle (say, an electron contained in an atom) drawn back to its position of equilibrium, O, by an 'elastic' force, *i.e.*

$$\ddot{\mathbf{r}} = -a^2\mathbf{r},$$

where a^2 is a positive scalar. Investigate the properties of the orbit for any given initial state, $\mathbf{r}=\mathbf{r}_0$, $\dot{\mathbf{r}}=\mathbf{v}_0$, for $t=0$. Write down the principle of areas and that of *vis-viva* and interpret them.

Elliptic harmonic motion. See 25.

46. Work out, vectorially, the elementary theory of the *Zeeman-effect*, that is to say, integrate the equation of motion

$$\ddot{\mathbf{r}} = -a^2\mathbf{r} - b^2\mathrm{V}\mathbf{v}\mathbf{M},$$

where $\mathbf{M}$ is a vector constant in space and time (homogeneous magnetic field), b^2 a positive scalar (the specific charge of an electron, e/m, divided by the velocity of light in vacuo), and $\mathbf{v}=\dot{\mathbf{r}}$.

Consider the projections of the orbit 1° on a plane normal to $\mathbf{M}$, and 2° on a plane parallel to $\mathbf{M}$. Compare the periods with that corresponding to the above example.

47. Show directly that the position of the centre of gravity (mass-centre), *i.e.* of the end-point of the vector $\mathbf{S}=\Sigma m\mathbf{r}/\Sigma m$, (17), Chap. III., is independent of the choice of any auxiliary framework.

48. Find the mass-centre of three equal masses placed at the corners of a triangle.

49. Find the mass-centre of four equal masses placed at the corners of a tetrahedron.

50. If $\mathbf{M}$ be the momentum of a system ($\Sigma m\mathbf{v}$), $\mathbf{q}$ its moment of momentum about O and $\mathbf{q}'$ that about O', show that $\mathbf{q}'=\mathbf{q}+\mathrm{V}\mathbf{a}\mathbf{M}$, where $\mathbf{a}$ is the vector drawn from O' to O. Interpret this simple property dynamically.

51. Prove that the resultant of two angular velocities, $\mathbf{p}_1$, $\mathbf{p}_2$, and hence also of infinitesimal rotations, of a rigid body, round axes which meet one another, is given by the vector $\mathbf{p}_1+\mathbf{p}_2$.

First show, by elementary considerations, that the velocity $\mathbf{v}$ of any point P of the rigid body is expressed by $\mathbf{v}=\mathrm{V}\mathbf{p}\mathbf{r}$, where $\mathbf{r}$ is the vector to P from any point O on the axis of rotation. Then, using the additive property of velocities, write $\mathbf{v}=\mathbf{v}_1+\mathbf{v}_2=\mathrm{V}\mathbf{p}_1\mathbf{r}+\mathrm{V}\mathbf{p}_2\mathbf{r}$, and so on.

52. Show that by any parallel shifting of the axis of rotation only a translation-velocity, of the body as a whole, is added.

Instead of $\mathbf{r}$ take $\mathbf{r}'=\mathbf{r}+\mathbf{a}$, and so on. Notice that if $\mathbf{a}\parallel\mathbf{p}$, the supplementary translational velocity vanishes; in other words, the shifting of $\mathbf{p}$ in its own line is a matter of indifference. Thus, $\mathbf{p}$, the angular velocity or '*spin*,' if taken with all of its kinematical implications, is a vector *localised* in a (straight) line, or a **rotor**. But if we are concerned with rotational motions only, this localisation becomes, of course, unessential, and spins can be manipulated with as free or 'unlocalised' vectors. There is, at any rate, no danger of misunderstanding arising from the above peculiarities. Another example of a vector localised in a line is a force applied to a rigid body.

53. Prove that the most general instantaneous motion of a rigid body is a screw motion or **twist** about a certain screw, *i.e.* an angular velocity $\mathbf{p}$ about a certain axis, combined with a translation velocity $\mathbf{v}_0$ along that axis.*

First write $\mathbf{v}=\mathbf{v}_0'+\mathrm{V}\mathbf{p}\mathbf{r}'$, where $\mathbf{v}_0'$ and $\mathbf{p}$ will be generally oblique; then decompose $\mathbf{v}_0'$ into components parallel and normal to $\mathbf{p}$, and so on.

54. Show directly that $\mathrm{div}\,\mathbf{v}=0$, thus verifying the incompressibility (Chap. V.) of a rigid body.

Write $\mathbf{v}=\mathbf{v}_0+\mathrm{V}\mathbf{p}\mathbf{r}$. Use for *div* the Hamiltonian applied scalarly.

55. The meaning of $\mathbf{v}$ being as above, prove directly that $\mathrm{curl}\,\mathbf{v}=2\mathbf{p}$.

56. Show that, for the most general motion of a rigid body,

$$\nabla(v^2)=\mathrm{V}\mathbf{v}\,\mathrm{curl}\,\mathbf{v}.$$

Also, prove that

$$(\mathbf{v}\nabla)\mathbf{v}=\mathrm{V}\mathbf{p}\mathbf{v},$$

so that

$$\frac{d\mathbf{v}}{dt}=\frac{\partial\mathbf{v}}{\partial t}+\mathrm{V}\mathbf{p}\mathbf{v}.$$ [Notation as in (74), Chap. V.]

Interpret the last result.

Cf. (22), Chap. IV.

57. In a rigid body (or system), let the mass be distributed symmetrically round an axis $\mathbf{a}$. Show that the inertial operator of the body, about a point on $\mathbf{a}$, is of the form

$$K=s_1-s_2\,\mathbf{a}.\mathbf{a},$$

where s_1, s_2 are positive scalars.

*The ratio $v_0:p$ is called the **pitch** of the screw. A quantity which, like a twist-velocity, has 'magnitude, direction, position [in a line], and pitch' has been called by Clifford a **motor**.

The above K' is written as a dyadic, the dot being a separator. Thus, if $\mathbf{q}$ be the moment of momentum and $\mathbf{p}$ the angular velocity, $\mathbf{q}=s_1\mathbf{p}-s_2\mathbf{a}(\mathbf{ap})$. Determine the values of s_1, s_2 in a few simple cases, such as a pair of point-masses and a massive (homogeneous) circle.

58. Let $\omega=\mathbf{i}\,.\,\mathbf{O}_1+\mathbf{j}\,.\,\mathbf{O}_2+\mathbf{k}\,.\,\mathbf{O}_3$ be any linear vector operator, as in (XIV.), Chap. V. Show directly that its principal values n are roots of the cubic equation

$$(\mathbf{O}_1-n\mathbf{i})\mathrm{V}(\mathbf{O}_2-n\mathbf{j})(\mathbf{O}_3-n\mathbf{k})=0.*$$

Develop this, by vector algebra, into the form $n^3-c_2n^2+c_1n-c_0=0$, and interpret this result.

Show that $\omega_{11}+\omega_{22}+\omega_{33}$, and so on, are invariants of the operator, *i.e.* independent of the orientation of the framework $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. Discuss as far as you can the general case and then the case of a symmetrical operator.

59. Using (XIII.), prove (XVI.'), Chap. V., *i.e.* that for any pair of vectors $\mathbf{A}$, $\mathbf{B}$,

$$\mathbf{B}\omega\mathbf{A}=\mathbf{A}\omega'\mathbf{B},$$

where ω is any linear vector operator and ω' its conjugate.

Show first that if $\omega=\Omega+\frac{1}{2}\mathrm{V}\epsilon$, then $\omega'=\Omega-\frac{1}{2}\mathrm{V}\epsilon$, and so on.

60. Investigate the properties of an operator ω, whose inverse is identical with its conjugate, $\omega'=\omega^{-1}$.

Consider the square of $\omega\mathbf{r}$, use (XVI.'), and so on.

61. Prove that every vector-field $\mathbf{w}$ can be decomposed, in one way only, into a purely irrotational and a purely solenoidal (sourceless) field.

First show that a field is *uniquely* determined by its sources (or sinks), its vortices and discontinuities;† hence, and so on. If the field in question be contained in a finite portion of space limited by σ, consider $\mathbf{w}_n$ as prescribed for every point of this surface; otherwise prescribe, in the usual way, the behaviour of $\mathbf{w}$ 'at infinity.' Try to do without potentials.

* This is immediately seen to be the equivalent of the usual determinant form

$$\begin{vmatrix} \omega_{11}-n & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22}-n & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33}-n \end{vmatrix}=0.$$

But do not resort to this form.

† Though discontinuities of $\mathbf{w}$ need not be mentioned expressly, their tangential parts being implied in the vortices and their normal parts in the sources and sinks (div).

62. Apply the general formula (75), Chap. V., to the case of a rigid body, developing, as it were, its differential kinematics. Show that, in Eulerian coordinates,

$$\frac{\partial v_1}{\partial x}=\frac{\partial v_2}{\partial y}=\frac{\partial v_3}{\partial z}=0, \quad \frac{\partial v_2}{\partial z}+\frac{\partial v_3}{\partial y}=0, \text{ etc.}$$

Write also the six corresponding differential conditions of rigidity in Lagrangian coordinates (a_1, a_2, a_3, t), using the displacement $\mathbf{D}$, or the vector $\mathbf{r}$.

Do not presuppose, of course, that $\mathbf{v}=\mathbf{v}_0+\mathrm{V}\boldsymbol{\rho}\mathbf{r}$. Use the immediate differential expression of rigidity, $\mathbf{l}\,d\mathbf{l}/dt=0$, for any line-element $\mathbf{l}$, and so on. The above is an instructive exercise by itself, and also in connexion with the theory of the relativistically rigid body, which theory is differential by necessity.

63. Let $\mathbf{R}$ be any vector-field extending in a moving deformable medium; $d\sigma$ being a surface element, composed always of the same particles of the medium, show that the time rate of change of the 'induction' of $\mathbf{R}$ through $d\sigma$, *i.e.* of $\mathbf{R}\mathbf{n}\,d\sigma$, is given by the formula

$$\frac{d}{dt}(\mathbf{R}\mathbf{n}\,d\sigma)=\mathbf{g}\mathbf{n}\,.\,d\sigma,$$

where

$$\mathbf{g}=\frac{\partial\mathbf{R}}{\partial t}+\mathbf{v}\,\mathrm{div}\,\mathbf{R}+\mathrm{curl}\,\mathrm{V}\mathbf{R}\mathbf{v},$$

$\partial/\partial t$ being the local time rate of change and $\mathbf{v}$ the velocity of the medium at the place considered.

Take for $d\sigma$ an infinitesimal parallelogram constructed on two individual line-elements $\mathbf{l}_1$, $\mathbf{l}_2$; then the induction through it will be $\mathbf{R}\mathrm{V}\mathbf{l}_1\mathbf{l}_2$. Differentiate this product, apply formula (75), Chap. V., and so on.

64. Similarly, $d\mathbf{s}$ being an individual line-element, show that

$$\frac{d}{dt}(\mathbf{R}\,d\mathbf{s})=\mathbf{h}\,d\mathbf{s},$$

where

$$\mathbf{h}=\frac{\partial\mathbf{R}}{\partial t}+\nabla(\mathbf{R}\mathbf{v})-\mathrm{V}\mathbf{v}\,\mathrm{curl}\,\mathbf{R}.$$

65. Maxwell's **electromagnetic stress** for the 'free aether' is given by the stress-operator,

$$p=u-\mathbf{E}\,.\,\mathbf{E}-\mathbf{M}\,.\,\mathbf{M},$$

where $\mathbf{E}$ is the electric, $\mathbf{M}$ the magnetic force and $u=\frac{1}{2}(E^2+M^2)$ the density of energy (pressures proper being counted positive,

tensions negative). Find the principal axes of this stress and the principal pressures or tensions. Write down the resultant force, per unit volume, $-\mathbf{i}\,\mathrm{div}\,\mathbf{p}_1 - \ldots$, assuming $\mathrm{div}\,\mathbf{E} = \mathrm{div}\,\mathbf{M} = 0$.

The principal pressures will be found equal u, $\pm\{u^2 - (\mathbf{VEM})^2\}^{\frac{1}{2}}$. Treat the general case of oblique $\mathbf{E}$, $\mathbf{M}$, then the simpler case of $\mathbf{E} \perp \mathbf{M}$, and finally the simplest case of purely electric, or purely magnetic, stress. Remember that the meaning of the above formula is

$$\mathbf{p}_n = u\mathbf{n} - \mathbf{E}(\mathbf{En}) - \mathbf{M}(\mathbf{Mn})$$

for any $\mathbf{n}$. Notice that this stress is self-conjugate or irrotational.

66. Write down the equation of energy for any individual portion of a deformable body, and interpret the various terms.

Use the general equation of motion (87), and proceed, *mutatis mutandis*, as in the simpler case of isotropic pressure (Chap. VI. p. 144). Express the work absorbed by the body on its being deformed in terms of elongations and shears, and normal and tangential pressures.

67. Show that the vector potential (114), Chap. VI., satisfies the solenoidal condition, $\mathrm{div}\,\mathbf{B} = 0$.

68. Let an incompressible liquid contain any number $\mathfrak{n}$ of closed vortex filaments s_ι, whose moments are $\mu_\iota (\iota = 1, 2, \ldots n)$. Let S be the 'current' through s_ι, *i.e.*

$$S_\iota = \rho \int \mathbf{v}\mathbf{n}_\iota \, d\sigma_\iota,$$

where σ_ι is a surface bounded by s_ι. Show that the kinetic energy of the liquid is equal to the sum of the products of the moments of the vortices into the currents through them,

$$\mathcal{L} = \sum_\iota^n \mu_\iota S_\iota.$$

Decompose the whole liquid into infinitesimal tubes of flow. An element of length $d\mathbf{s}$ of one of them, of cross section $d\sigma$, will contain the energy $\frac{1}{2}\rho v^2 . d\tau = \frac{1}{2}\rho v\, d\sigma \,.\, \mathbf{v}\, d\mathbf{s}$. Remember that, the motion being solenoidal, $\rho v\, d\sigma$ is constant along the tube. Integrate the above expression along the tube, and so on. At a certain stage of your reasoning consider separately the case of a liquid enclosed in a rigid shell and that of an unlimited liquid.

69. Show that the above 'currents' are linear homogeneous functions of the moments of all vortex filaments,

$$S_\kappa = \sum_{\iota=1}^{n} c_{\kappa\iota}\mu_\iota, \quad \kappa = 1, 2, \ldots n,$$

with coefficients depending only on the size, shape and relative position of the vortices. Whence represent the kinetic energy of the liquid as a homogeneous quadratic function of the moments. Determine also the coefficients $c_{\kappa\iota}$, thus obtaining again the formula (117), Chap. VI.

To determine the coefficients $c_{\kappa\iota} = c_{\iota\kappa}$, transform the currents through s_κ into line-integrals round s_κ, using the vector potential $\mathbf{B}$. Notice the analogy between the above expressions for the kinetic energy of a liquid and those for the magnetic or 'electro-kinetic' energy of a system of electric-current circuits, the liquid currents S_κ corresponding to the inductions through the circuits, the moments μ_κ to the electric-current intensities, $c_{\iota\iota}$, $c_{\iota\kappa}$ to the coefficients of self- and mutual induction. Remember, however, that these formulae are neither characteristic of liquid motion nor of the magnetic field of electrical currents. They are valid for any solenoidal vector-field.

CARTESIAN EQUIVALENTS OF VECTOR FORMULAE.

In order to make clear to the majority of readers, brought up in familiarity with Cartesian components, both the advantage and the meaning of the above vector treatment, we shall give here the Cartesian equivalents of nearly all the vector formulae and symbols contained in Chapter I. and in the chapters on Mechanics, II.-VI., or, as it were, a kind of dictionary to the whole book.

The Cartesian, *i.e.* rectangular components of any vector $\mathbf{A}$, will be denoted throughout by A_1, A_2, A_3, with the one exception of the coordinates of a point relatively to a fixed system of reference, which (suiting the general use) will be denoted by x, y, z.

It will be convenient to divide the whole dictionary into two parts, the first containing *purely mathematical* and the second comprehending *chiefly mechanical* formulae.

FIRST PART.

Vector sum:

$\mathbf{R} = \mathbf{A} \pm \mathbf{B} = \pm \mathbf{B} + \mathbf{A}$. | $R_1 = A_1 \pm B_1$, $R_2 = A_2 \pm B_2$, $R_3 = A_3 \pm 1$

Scalar product:

$\mathbf{AB} = \mathbf{BA}$. | $A_1B_1 + A_2B_2 + A_3B_3$.

Vector product:

$\mathbf{C} = \mathrm{V}\mathbf{AB} = -\mathrm{V}\mathbf{BA}$. | $C_1 = A_2B_3 - A_3B_2$, $C_2 = A_3B_1 - A_1B_3$
$C_3 = A_1B_2 - A_2B_1$.

$\mathbf{AVBC} = \mathbf{BVCA} = \mathbf{CVAB}$. | $A_1(B_2C_3 - B_3C_2) + \ldots = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$

$\mathbf{R} = \mathrm{V}\mathbf{A}(\mathbf{B} \pm \mathbf{C})$. | $R_1 = A_2(B_3 \pm C_3) - A_3(B_2 \pm C_2)$,
etc., by cyclic permutation of 1, 2, 3.

$\mathbf{S} = \mathrm{V}\mathbf{A}\mathrm{V}\mathbf{BC}$
$= (\mathbf{AC})\mathbf{B} - (\mathbf{AB})\mathbf{C}$. | $S_1 = (A_1C_1 + A_2C_2 + A_3C_3)B_1$
$- (A_1B_1 + A_2B_2 + A_3B_3)C_1$,
etc., by cyclic permutation of 1, 2, 3.

$\dot{\mathbf{A}} = \dfrac{d\mathbf{A}}{dt}$. | $\dot{A}_1 = dA_1/dt$, $\dot{A}_2 = dA_2/dt$, $\dot{A}_3 = dA_3/dt$,
supposing *fixed* Cartesian axes.

Line integral:

$I = \int \mathbf{R}\, ds$. | $I = \int (R_1 dx + R_2 dy + R_3 dz)$
$= \int \left(R_1 \dfrac{dx}{ds} + \ldots \right) ds$.

Surface-integral:

$S = \int \mathbf{Rn}\, d\sigma$. | $S = \int (R_1 n_1 + R_2 n_2 + R_3 n_3)\, d\sigma$,
n_1, n_2, n_3 being the direction-cosines of $\mathbf{n}$.

Curl or *Rotation*:

$\mathbf{C} = \mathrm{curl}\,\mathbf{R} = \mathrm{V}\nabla\mathbf{R}$. | $C_1 = \dfrac{\partial R_3}{\partial y} - \dfrac{\partial R_2}{\partial z}$, $C_2 = \dfrac{\partial R_1}{\partial z} - \dfrac{\partial R_3}{\partial x}$,
$C_3 = \dfrac{\partial R_2}{\partial x} - \dfrac{\partial R_1}{\partial y}$.

Divergence:

$\mathrm{div}\,\mathbf{R} = \nabla\mathbf{R}$. | $\dfrac{\partial R_1}{\partial x} + \dfrac{\partial R_2}{\partial y} + \dfrac{\partial R_3}{\partial z}$.

Theorem of Divergence, called also Gauss' Theorem:

$\int \mathrm{div}\,\mathbf{R}\, d\tau = \int \mathbf{Rn}\, d\sigma$, | $\int \left(\dfrac{\partial R_1}{\partial x} + \dfrac{\partial R_2}{\partial y} + \dfrac{\partial R_3}{\partial z} \right) d\tau$
$= \int (R_1 n_1 + R_2 n_2 + R_3 n_3)\, d\sigma$.

$\mathbf{n}$ being the *outward* normal.

Slope or *gradient*:

$\mathbf{R} = \nabla\phi$, | $R_1 = \dfrac{\partial \phi}{\partial x}$, $R_2 = \dfrac{\partial \phi}{\partial y}$, $R_3 = \dfrac{\partial \phi}{\partial z}$.

ϕ being a *scalar* function.

Stokes' Theorem:

$\int \mathbf{n}\, \mathrm{curl}\,\mathbf{R}\, d\sigma = \int_{(s)} \mathbf{R}\, ds$. | $\int \left\{ n_1 \left(\dfrac{\partial R_3}{\partial y} - \dfrac{\partial R_2}{\partial z} \right) + n_2 \left(\dfrac{\partial R_1}{\partial z} - \dfrac{\partial R_3}{\partial x} \right) \right.$
$\left. + n_3 \left(\dfrac{\partial R_2}{\partial x} - \dfrac{\partial R_1}{\partial y} \right) \right\} d\sigma$
$= \int_{(s)} (R_1 dx + R_2 dy + R_3 dz)$.

Formula (25), Chap. I. (cited also under (xx.) in Chap. VI.):

$$\operatorname{div} V\mathbf{RS} = \mathbf{S}\operatorname{curl}\mathbf{R} - \mathbf{R}\operatorname{curl}\mathbf{S}.$$

$$\frac{\partial}{\partial x}(R_2S_3 - R_3S_2) + \text{etc.}$$
$$= S_1\left(\frac{\partial R_3}{\partial y} - \frac{\partial R_2}{\partial z}\right) + \text{etc.}$$
$$- R_1\left(\frac{\partial S_3}{\partial y} - \frac{\partial S_2}{\partial z}\right) - \text{etc.},$$

'etc.' meaning always two other terms obtained from the preceding by cyclic permutation of 1, 2, 3 and of x, y, z.

Formula (xxi.), Chap. VI.:

$$\operatorname{curl} V\mathbf{wv} = \mathbf{w}\operatorname{div}\mathbf{v} + (\mathbf{v}\nabla)\mathbf{w} - \mathbf{v}\operatorname{div}\mathbf{w} - (\mathbf{w}\nabla)\mathbf{v}.$$

$$\frac{\partial}{\partial y}(w_1v_2 - w_2v_1) - \frac{\partial}{\partial z}(w_3v_1 - w_1v_3)$$
$$= w_1\left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right)$$
$$+ \left(v_1\frac{\partial}{\partial x} + v_2\frac{\partial}{\partial y} + v_3\frac{\partial}{\partial z}\right)w_1$$
$$- v_1\left(\frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y} + \frac{\partial w_3}{\partial z}\right)$$
$$- \left(w_1\frac{\partial}{\partial x} + w_2\frac{\partial}{\partial y} + w_3\frac{\partial}{\partial z}\right)v_1,$$

with two similar equations.

Axial differentiation:

$$\mathbf{s}\nabla.$$

$$s_1\frac{\partial}{\partial x} + s_2\frac{\partial}{\partial y} + s_3\frac{\partial}{\partial z} = \frac{\partial}{\partial s}.$$

Laplacian, applied to a *scalar*:

$$\nabla^2\phi = \nabla\nabla\phi = \operatorname{div}\nabla\phi.$$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}.$$

Laplacian, applied to a *vector*:

$$\mathbf{S} = \nabla^2\mathbf{R}.$$

$$S_1 = \nabla^2R_1, \quad S_2 = \nabla^2R_2, \quad S_3 = \nabla^2R_3,$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$$

as above.

Green's Theorem:

$$\int\phi\nabla^2\psi\,d\tau = \int\phi\frac{\partial\psi}{\partial n}\,d\sigma - \int\nabla\phi\,.\,\nabla\psi\,.\,d\tau;$$

$$\int\phi\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right)d\tau = \int\phi\frac{\partial\psi}{\partial n}\,d\sigma - \int\left\{\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial y} + \frac{\partial\phi}{\partial z}\frac{\partial\psi}{\partial z}\right\}d\tau.$$

to obtain it, put in the above Theorem of Divergence

$$\mathbf{R} = \phi\nabla\psi,$$

and consequently

$$\operatorname{div}\mathbf{R} = \nabla\phi\,.\,\nabla\psi + \phi\nabla^2\psi,$$
$$\mathbf{Rn} = \phi\frac{\partial\psi}{\partial n},$$

ϕ, ψ being *scalar* functions.*

Iteration of curl, (33), Chap. I.:

$$\mathbf{S} = \operatorname{curl}^2\mathbf{R} = \nabla\operatorname{div}\mathbf{R} - \nabla^2\mathbf{R}.$$

$$S_1 = \frac{\partial}{\partial x}\left(\frac{\partial R_1}{\partial x} + \frac{\partial R_2}{\partial y} + \frac{\partial R_3}{\partial z}\right) - \left(\frac{\partial^2R_1}{\partial x^2} + \frac{\partial^2R_1}{\partial y^2} + \frac{\partial^2R_1}{\partial z^2}\right),$$

etc.

Linear vector-operator, (xiii.), Chap. V.:

$$\omega\mathbf{A} = (\Omega + \tfrac{1}{2}V\mathbf{c})\mathbf{A},$$
$$\text{say} = \mathbf{B}.$$

$$B_1 = \Omega_1A_1 + \tfrac{1}{2}(c_2A_3 - c_3A_2),$$

etc.

the components A_1, B_1, etc., being taken along the *principal axes* of the symmetrical part Ω of the operator ω.

SECOND PART.

D'Alembert's Principle:

$$\Sigma(m\ddot{\mathbf{r}} - \mathbf{F})\delta\mathbf{r} = 0.$$

$$\Sigma\{(m\ddot{x} - F_1)\delta x + (m\ddot{y} - F_2)\delta y + (m\ddot{z} - F_3)\delta z\} = 0.$$

Lagrange's Equations of Motion:

$$m_i\ddot{\mathbf{r}}_i = \mathbf{F}_i + \lambda\nabla_{(i)}\phi + \mu\nabla_{(i)}\psi + \dots;$$

n vector equations.

$$m_i\ddot{x}_i = F_{i1} + \lambda\frac{\partial\phi}{\partial x_i} + \mu\frac{\partial\psi}{\partial x_i} + \dots,$$
$$m_i\ddot{y}_i = F_{i2} + \lambda\frac{\partial\phi}{\partial y_i} + \mu\frac{\partial\psi}{\partial y_i} + \dots,$$
$$m_i\ddot{z}_i = F_{i3} + \lambda\frac{\partial\phi}{\partial z_i} + \mu\frac{\partial\psi}{\partial z_i} + \dots;$$

$3n$ scalar equations.

*To obtain the particular form of Green's Theorem applied on p. 155, put $\psi = \phi$, and consequently $\nabla\phi\,.\,\nabla\psi = (\nabla\phi)^2$.

Hamilton's Principle, (12), Chap. II., has nothing vectorial about it. Thus, it will be sufficient to write only the right-hand side of equation (11), from which the principle followed, along with its Cartesian equivalent:

$$\left[\Sigma m\mathbf{v}\,\delta\mathbf{r}\right]_a^b. \qquad \left[\Sigma m(v_1\,\delta x + v_2\,\delta y + v_3\,\delta z)\right]_a^b.$$

Then, continuing our survey, we have the following equivalences:

Centre of mass, (17), Chap. III.:

$$\mathbf{S} = \Sigma m\mathbf{r}/M, \quad \text{where } M = \Sigma m.$$

$$S_1 = \frac{1}{M}\Sigma mx, \quad S_2 = \frac{1}{M}\Sigma my, \quad S_3 = \frac{1}{M}\Sigma mz,$$

S_1, etc., being simply the Cartesian co-ordinates of the centre of mass.

The resultant *moment* of impressed forces:

$$\mathbf{L} = \Sigma \mathrm{V}\mathbf{r}\mathbf{F}.$$

$$L_1 = \Sigma(yF_3 - zF_2), \text{ etc.}$$

The *principle of areas*, (20), Chap. III.:

$$\frac{d}{dt}\Sigma m\mathrm{V}\mathbf{r}\mathbf{v} = \mathbf{L}.$$

$$\frac{d}{dt}\Sigma m(yv_3 - zv_2) = L_1, \text{ etc.}$$

Fundamental kinematical relations for a *rigid* system, (22), Chap. IV.:

$$\dot{\mathbf{w}} = \dot{\mathbf{w}}' + \mathrm{V}\mathbf{p}\mathbf{w}.$$

$$\frac{dw_1}{dt} = \frac{dw_1'}{dt} + p_2w_3 - p_3w_2, \text{ etc.},$$

w_1, w_2, w_3 being the components of $\mathbf{w}$ along fixed axes, and w_1', etc., the components of the same along axes moving with the rigid system.*

Moment of momentum, (24), Chap. IV.:

$$\mathbf{q} = \Sigma m\mathrm{V}\mathbf{r}\mathbf{v}.$$

$$q_1 = \Sigma m(yv_3 - zv_2), \text{ etc.}$$

Equations of motion of a *rigid* system, (25), (26), *ibid.*:

$$\frac{d\mathbf{q}}{dt} = \frac{d\mathbf{q}'}{dt} - \mathrm{V}\mathbf{q}\mathbf{p} = \mathbf{L}.$$

$$\frac{dq_1}{dt} = \frac{dq_1'}{dt} - (q_2p_3 - q_3p_2) = L_1, \text{ etc.}$$

* Remember that p_1, p_2, p_3, the components of the angular velocity $\mathbf{p}$, have been taken always along the moving axes, namely along the principal axes of the rigid system. (Chap. IV.)

Kinetic energy, (27), (27*a*):

$$T = \tfrac{1}{2}\mathbf{p}\mathbf{q} = \tfrac{1}{2}\mathbf{p}K\mathbf{p}.$$

$$T = \tfrac{1}{2}(p_1q_1 + p_2q_2 + p_3q_3) = \tfrac{1}{2}(K_1p_1^2 + K_2p_2^2 + K_3p_3^2),$$

K_1, etc., being the principal values of the (symmetrical) linear operator K.

Euler's eq. of motion, (26*a*):

$$K\frac{d\mathbf{p}'}{dt} = \mathbf{L} - \mathrm{V}\mathbf{p}K\mathbf{p}.$$

$$K_1\frac{dp_1}{dt} = L_1 + (K_2 - K_3)p_2p_3, \text{ etc.},$$

where the accents could be omitted, since p_1, etc., are already taken along the (moving) principal axes.

Time as a function of angular velocity, formula (*a*), Chap. IV., page 82:

$$t = \int_{p_0}^{p} \frac{p\,dp}{N}, \qquad (a)$$

where

$$N = \mathbf{p}\mathrm{V}(K^{-1}\mathbf{p})(K\mathbf{p}).$$

To obtain the usual, Cartesian, development of this denominator, apply the determinantal form of the product $\mathbf{AVBC}$ (see *First Part* of this 'Appendix'); then

$$N = \begin{vmatrix} p_1 & p_2 & p_3 \\ p_1/K_1 & p_2/K_2 & p_3/K_3 \\ K_1p_1 & K_2p_2 & K_3p_3 \end{vmatrix} = p_1p_2p_3\Delta,$$

where

$$\Delta = \frac{K_2 - K_3}{K_1} + \frac{K_3 - K_1}{K_2} + \frac{K_1 - K_2}{K_3}.$$

Now, p_1, p_2, p_3 may readily be expressed by p, and by the invariants $2T$ and q, namely by solving the three equations,

$$p_1^2 + p_2^2 + p_3^2 = p^2, \quad K_1p_1^2 + K_2p_2^2 + K_3p_3^2 = 2T,$$
$$K_1^2p_1^2 + K_2^2p_2^2 + K_3^2p_3^2 = q^2.$$

Thus, writing

$$\Delta' = \begin{vmatrix} 1 & 1 & 1 \\ K_1 & K_2 & K_3 \\ K_1^2 & K_2^2 & K_3^2 \end{vmatrix},$$

we get immediately

$$p_1^2 = \frac{K_2K_3(K_3 - K_2)}{\Delta'}\left\{p^2 - \frac{2T(K_2 + K_3) - q^2}{K_2K_3}\right\}, \text{ etc.},$$

whence

$$N = \frac{\Delta}{\Delta'^{\frac{1}{2}}}\sqrt{(C_1p^2 - \lambda_1)(C_2p^2 - \lambda_2)(C_3p^2 - \lambda_3)},$$

which is the usual form given in text-books on rigid dynamics, the constants under the radical being given by

$$C_1=(K_3-K_2)K_2K_3, \text{ etc.};$$

$$\lambda_1=(K_3-K_2)\{2T(K_2+K_3)-q^2\}, \text{ etc.}$$

Introducing the last expression of N into the above integral (α), we get what is alluded to in Chap. IV. p. 82.

But let us continue the columnar system of our Vectorially-Cartesian dictionary.

Fundamental formula of strain-theory, (48), Chap. V.:

$$\mathbf{l}'=\mathbf{l}+(\mathbf{l}\nabla)\mathbf{D}.$$

$$l_1'=l_1+\left(l_1\frac{\partial}{\partial x}+l_2\frac{\partial}{\partial y}+l_3\frac{\partial}{\partial z}\right)D_1, \text{ etc.},$$

where x, y, z are the coordinates in the unstrained body, identical with our a_1, a_2, a_3; l_1, l_1', etc., are the projections of $\mathbf{l}$, $\mathbf{l}'$ on the axes of x, y, z.

Developed form of the above, (50), (51):

$$\mathbf{l}'=\Omega\mathbf{l}-\tfrac{1}{2}V\mathbf{l}\operatorname{curl}\mathbf{D}.$$

$$l_1'=\Omega_1 l_1-\tfrac{1}{2}\left\{l_2\left(\frac{\partial D_2}{\partial x}-\frac{\partial D_1}{\partial y}\right)-l_3\left(\frac{\partial D_1}{\partial z}-\frac{\partial D_3}{\partial x}\right)\right\}, \text{ etc.},$$

where x, y, z and the components of $\mathbf{l}$, $\mathbf{l}'$, $\mathbf{D}$ are taken along the principal axes of Ω at the point x, y, z.

Cubic dilatation, (59):

$$\theta=\operatorname{div}\mathbf{D}.$$

$$\theta=\frac{\partial D_1}{\partial x}+\frac{\partial D_2}{\partial y}+\frac{\partial D_3}{\partial z}.$$

Rotation (infinitesimal):

$$\mathbf{c}=\tfrac{1}{2}\operatorname{curl}\mathbf{D}.$$

$$c_1=\tfrac{1}{2}\left(\frac{\partial D_3}{\partial y}-\frac{\partial D_2}{\partial z}\right), \text{ etc.}$$

Equation of continuity, (60):

$$\delta\rho+\rho\operatorname{div}\mathbf{D}=0.$$

$$\delta\rho+\rho\left(\frac{\partial D_1}{\partial x}+\frac{\partial D_2}{\partial y}+\frac{\partial D_3}{\partial z}\right).$$

Surface of discontinuity; identical conditions, (71):

$$[\nabla(\mathbf{Dx})]=\mathbf{n}(\mathbf{mx});$$

put $\mathbf{x}=\mathbf{i}, \mathbf{j}, \mathbf{k}$ respectively.

$$\left[\frac{\partial D_1}{\partial x}\right]=m_1n_1, \quad \left[\frac{\partial D_2}{\partial x}\right]=m_2n_1, \quad \left[\frac{\partial D_3}{\partial x}\right]=m_3n_1,$$

$$\left[\frac{\partial D_1}{\partial y}\right]=m_1n_2, \quad \left[\frac{\partial D_2}{\partial y}\right]=m_2n_2, \quad \left[\frac{\partial D_3}{\partial y}\right]=m_3n_2,$$

$$\left[\frac{\partial D_1}{\partial z}\right]=m_1n_3, \quad \left[\frac{\partial D_2}{\partial z}\right]=m_2n_3, \quad \left[\frac{\partial D_3}{\partial z}\right]=m_3n_3;$$

these are the *nine* scalar conditions of Hadamard.

Idem, (72), (73):

$$[\theta]=[\operatorname{div}\mathbf{D}]=\mathbf{nm},$$

$$[\mathbf{c}]=\tfrac{1}{2}[\operatorname{curl}\mathbf{D}]=\tfrac{1}{2}V\mathbf{nm}.$$

$$[\theta]=n_1m_1+n_2m_2+n_3m_3,$$

$$[c_1]=\tfrac{1}{2}(n_2m_3-n_3m_2), \text{ etc.}$$

Individual and local time-variation, (74):

$$\frac{d}{dt}=\frac{\partial}{\partial t}+(\mathbf{v}\nabla).$$

$$\frac{d}{dt}=\frac{\partial}{\partial t}+v_1\frac{\partial}{\partial x}+v_2\frac{\partial}{\partial y}+v_3\frac{\partial}{\partial z}.$$

Variation of a line-element, (75):

$$\frac{d\mathbf{l}}{dt}=(\mathbf{l}\nabla)\mathbf{v}.$$

$$\frac{dl_1}{dt}=\left(l_1\frac{\partial}{\partial x}+l_2\frac{\partial}{\partial y}+l_3\frac{\partial}{\partial z}\right)v_1, \text{ etc.}$$

Variation of a volume-element, (77):

$$\frac{d}{dt}(d\tau)=\operatorname{div}\mathbf{v}\,.\,d\tau.$$

$$\frac{d}{dt}(d\tau)=\left(\frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial v_3}{\partial z}\right)d\tau.$$

Eq. of continuity, (78a):

$$\frac{\partial\rho}{\partial t}+\operatorname{div}(\rho\mathbf{v})=0.$$

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho v_1)+\frac{\partial}{\partial y}(\rho v_2)+\frac{\partial}{\partial z}(\rho v_3)=0.$$

Kinematical conditions of compatibility, (82):

$$[\mathbf{v}]=\left[\frac{d\mathbf{D}}{dt}\right]=-\mathbf{m}\mathfrak{v}. \qquad [v_1]=\left[\frac{dD_1}{dt}\right]=-m_1\mathfrak{v}, \text{ etc.}$$

General eq. of motion of a non-rigid body, (87):

$$\rho\frac{d\mathbf{v}}{dt}=\rho\mathbf{F}-\mathbf{i}\,\mathrm{div}\,\mathbf{p}_1-\mathbf{j}\,\mathrm{div}\,\mathbf{p}_2-\mathbf{k}\,\mathrm{div}\,\mathbf{p}_3$$
$$\equiv\rho\mathbf{F}-\mathrm{div}\{\mathbf{p}_1.\mathbf{i}+\mathbf{p}_2.\mathbf{j}+\mathbf{p}_3.\mathbf{k}\},$$

the dots being separators.

$$\rho\frac{dv_1}{dt}=\rho F_1-\left(\frac{\partial p_{11}}{\partial x}+\frac{\partial p_{12}}{\partial y}+\frac{\partial p_{13}}{\partial z}\right),$$
$$\rho\frac{dv_2}{dt}=\rho F_2-\left(\frac{\partial p_{21}}{\partial x}+\frac{\partial p_{22}}{\partial y}+\frac{\partial p_{23}}{\partial z}\right),$$
$$\rho\frac{dv_3}{dt}=\rho F_3-\left(\frac{\partial p_{31}}{\partial x}+\frac{\partial p_{32}}{\partial y}+\frac{\partial p_{33}}{\partial z}\right).$$

Hydrodynamical differential equation, (90), Chap. VI.:

$$\frac{d\mathbf{v}}{dt}\equiv\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\nabla)\mathbf{v}=\mathbf{F}-\frac{1}{\rho}\nabla p.$$

$$\frac{dv_1}{dt}\equiv\frac{\partial v_1}{\partial t}+\left(v_1\frac{\partial}{\partial x}+v_2\frac{\partial}{\partial y}+v_3\frac{\partial}{\partial z}\right)v_1$$
$$=F_1-\frac{1}{\rho}\frac{\partial p}{\partial x}, \text{ etc.}$$

[Equation of continuity, see above.]

Circulation, ibidem:

$$I=\int_{(s)}\mathbf{v}\,d\mathbf{s}. \qquad I=\int_{(s)}\left(v_1\frac{dx}{ds}+v_2\frac{dy}{ds}+v_3\frac{dz}{ds}\right)ds.$$

Hydrodynamical eq. *for conservative forces and* $\rho=f(p)$, (93*a*):

$$\frac{\partial\mathbf{v}}{\partial t}+2\mathrm{V}\mathbf{w}\mathbf{v}=\nabla\left(Q-\frac{v^2}{2}\right), \qquad \frac{\partial v_1}{\partial t}+2(w_2v_3-w_3v_2)=\frac{\partial}{\partial x}\left(Q-\frac{v^2}{2}\right), \text{ etc.,}$$

where $\mathbf{w}=\frac{1}{2}\,\mathrm{curl}\,\mathbf{v}$. where $w_1=\frac{1}{2}\left(\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z}\right)$, etc.

Clebsch's Transformation, (96), (97):

$$\mathbf{v}=\nabla\phi+\lambda\nabla\psi; \qquad v_1=\frac{\partial\phi}{\partial x}+\lambda\frac{\partial\psi}{\partial x}, \text{ etc.;}$$

$$\therefore\ \mathbf{w}=\tfrac{1}{2}\mathrm{V}\nabla\lambda.\nabla\psi. \qquad \therefore\ w_1=\tfrac{1}{2}\left(\frac{\partial\lambda}{\partial y}\frac{\partial\psi}{\partial z}-\frac{\partial\lambda}{\partial z}\frac{\partial\psi}{\partial y}\right), \text{ etc.}$$

Kinetic energy of vortex motion, (116), (117):

$$L=\rho\int\mathbf{B}\mathbf{w}\,d\tau \qquad L=\rho\int(B_1w_1+\ldots)d\tau$$
$$=\frac{\rho}{2\pi}\iint\frac{\mathbf{w}\mathbf{w}'}{r}d\tau\,d\tau'. \qquad =\frac{\rho}{2\pi}\iint\frac{w_1w_1'+\cdots}{r}d\tau\,d\tau'.$$

Helmholtz's eq. of vortex motion, (123):

$$\frac{d}{dt}\left(\frac{\mathbf{w}}{\rho}\right)=\frac{1}{\rho}(\mathbf{w}\nabla)\mathbf{v}. \qquad \frac{d}{dt}\left(\frac{w_1}{\rho}\right)=\frac{1}{\rho}\left(w_1\frac{\partial}{\partial x}+w_2\frac{\partial}{\partial y}+w_3\frac{\partial}{\partial z}\right)v_1, \text{ etc.}$$

Wave of acceleration, (130):

$$[\nabla\log\rho]=\frac{1}{\mathfrak{v}}(\mathbf{mn})\mathbf{n}. \qquad \left[\frac{\partial\log\rho}{\partial x}\right]=\frac{1}{\mathfrak{v}}(m_1n_1+m_2n_2+m_3n_3)n_1,$$
$$\left[\frac{\partial\log\rho}{\partial y}\right]=\frac{1}{\mathfrak{v}}(m_1n_1+m_2n_2+m_3n_3)n_2,$$
$$\left[\frac{\partial\log\rho}{\partial z}\right]=\frac{1}{\mathfrak{v}}(m_1n_1+m_2n_2+m_3n_3)n_3.$$

Idem, (131):

$$\mathfrak{v}^2\mathbf{m}=\frac{dp}{d\rho}(\mathbf{mn})\mathbf{n}, \qquad \mathfrak{v}^2m_1=\frac{dp}{d\rho}(m_1n_1+m_2n_2+m_3n_3)n_1, \text{ etc.,}$$

where $\mathbf{m}$ is the vector characterising the discontinuity, where

$$\mathbf{m}=-\frac{1}{\mathfrak{v}}\left[\frac{d\mathbf{v}}{dt}\right]. \qquad m_1=-\frac{1}{\mathfrak{v}}\left[\frac{dv_1}{dt}\right],$$
$$m_2=-\frac{1}{\mathfrak{v}}\left[\frac{dv_2}{dt}\right],$$
$$m_3=-\frac{1}{\mathfrak{v}}\left[\frac{dv_3}{dt}\right].$$

MISCELLANEOUS NOTES.

A. Projective Vector Equality.

THE definition of vector equality given on p. 3 is based, as is in fact the whole text of the book, upon the metrical concept of "length" or "distance" and on Euclidean parallelism. It may, however, be worth while to draw the reader's attention to the possibility of a non-metrical or *projective* definition of *the equality* of two vectors and to the utility of such a concept. This subject is treated fully in my *Projective Vector Algebra* (London, Bell, 1919), to which the reader may be referred for details and applications. Here it will be enough to explain the concept in question in a brief and somewhat modified exposition.

The usual axioms of projective, three-dimensional, geometry being assumed, introduce a fixed plane τ chosen conventionally. Call *vector* an ordered pair of points, O the origin and A the end-point, and refer to the intersection point of the straight line OA with the plane τ as "the terminus" of the vector. Then our definition can be worded as follows:

Two vectors OA and $O'A'$ are *equal* to each other if they are coterminal and if OO' and AA' are coterminal, that is to say, if the straight lines OA and $O'A'$, OO' and AA' meet on the τ-plane.

The usual definition is manifestly but a special case of this more general one.

The success of such a non-metrical definition of vector equality is mainly conditioned by its *transitivity*, and it is interesting to note that the latter property is a consequence of, or equivalent to, a special case of the theorem of Desargues about perspective triangles (cf. p. 171). In fact, let

$$OA = O'A' \quad \text{and} \quad O'A' = O''A'',$$

and let us ask whether these vector equalities entail also that of OA and $O''A''$. Now, the first and the second as well as the second

and the third vectors being coterminal, say in S (a point of the τ-plane), so also are the first and the third. Thus the triangles $OO'O''$ and $AA'A''$ are perspective from the point S, and will therefore, by Desargues' theorem, be also perspective from a line a on τ (Fig. 52). But this means that the lines OO'' and AA'' will also

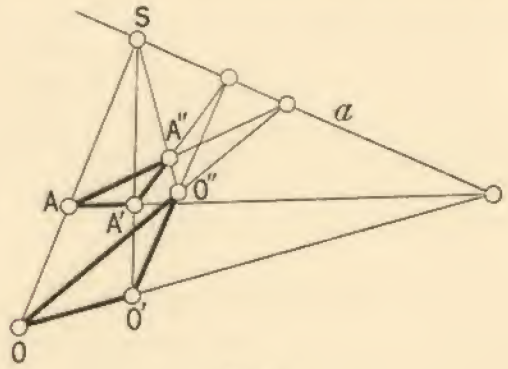

FIG. 52.

meet on τ, namely on the line a, so that $OA = O''A''$, the transitivity which was to be proved.

Notice that the Desargues theorem appears here in its special case, the point of perspectivity (S) being on the line of perspectivity (a) of the two triangles.

The transitivity of projective vector equality thus established can at once be utilized for the construction of equal vectors on the same straight line and, in particular, for the doubling of a given vector, etc., as shown in the book just quoted.

B. Various Notations of Products of Vectors.

The notation for a vector product of two vectors, VAB, introduced on p. 13 and used throughout the text, is due to Oliver Heaviside, who borrowed it from Hamilton's quaternionic symbolism. J. W. Gibbs, and after him E. B. Wilson and others, wrote $\mathbf{A} \times \mathbf{B}$ for the vector product and $\mathbf{A} \cdot \mathbf{B}$ for the scalar product of two vectors **A**, **B**. More recently, H. A. Lorentz and most modern vectorists write $[\mathbf{AB}]$ for the vectorial and $(\mathbf{AB})$ for the scalar product. The corresponding products of, say, **A** into $\mathbf{B}+\mathbf{C}$ are written $[\mathbf{A}, \mathbf{B}+\mathbf{C}]$ and $(\mathbf{A}, \mathbf{B}+\mathbf{C})$, which does not at all seem

convenient for quick reading. On the whole, Heaviside's notation, adopted in this book, has seemed the most convenient. But if some of my readers have made up their mind to familiarize themselves with any other of the current notations, I am the last man to dissuade them from doing so. Provided they keep always to *the same* notation, a skillful handling of the vector calculus will not fail to ensue.

C. (to p. 65). The Principle of Areas.

If a system is known to satisfy the principle of areas only for two perpendicular axes, then, as was pointed out by Jacobi (*Vorlesungen über Dynamik*, Berlin, 1866, p. 33), the principle holds also for a third axis perpendicular to those and, therefore, for any axis.

D. Linear Vector Operators.

The properties proved on p. 75 for the "inertial" operator K, and based on its special structure, can easily be shown to hold for any *symmetrical* linear vector operator and, apart from the *orthogonality* of the principal axes, for any linear vector operator whatever.

In fact, a linear vector operator ω can, in a most general way, be defined as a continuous vector function of the vector operand having the *distributive* property, *i.e.*

$$\omega(\mathbf{A}+\mathbf{B})=\omega\mathbf{A}+\omega\mathbf{B},$$

for any two vectors $\mathbf{A}$, $\mathbf{B}$. Whence, $\omega(2\mathbf{A})=2\omega\mathbf{A}$ and, by a simple reasoning, if n be any scalar,

$$\omega(n\mathbf{A})=n\omega\mathbf{A}.$$

It follows at once that an operator ω is completely determined if it is given for any three non-coplanar operands $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, that is to say, if the values

$$\omega\mathbf{a}=\mathbf{a}', \quad \omega\mathbf{b}=\mathbf{b}', \quad \omega\mathbf{c}=\mathbf{c}'$$

are prescribed. For since any vector $\mathbf{r}$ can be represented by

$$\mathbf{r}=x\mathbf{a}+y\mathbf{b}+z\mathbf{c},$$

where x, y, z are some scalars, we have

$$\mathbf{r}'=\omega\mathbf{r}=x\omega\mathbf{a}+y\omega\mathbf{b}+z\omega\mathbf{c}.$$

The three vector data can themselves be expanded, thus

$$\omega\mathbf{a}=\omega_{11}\mathbf{a}+\omega_{12}\mathbf{b}+\omega_{13}\mathbf{c}, \text{ etc.},$$

where ω_{ik}, in general different from ω_{ki}, are scalars. The most general linear vector operator is thus seen to imply nine independent scalar data.

Now, *the principal axes* of ω are defined as those directions of the operand $\mathbf{r}$ for which

$$\omega\mathbf{r}=n\mathbf{r},$$

n being a (positive) scalar factor, the corresponding "principal value" of ω. The required properties of the operator follow from this definition at once.

First, if $\mathbf{x}_1$ and $\mathbf{x}_2$ are two principal axes with equal values of n, we have, for any coplanar vector $\alpha\mathbf{x}_1+\beta\mathbf{x}_2$,

$$\omega(\alpha\mathbf{x}_1+\beta\mathbf{x}_2)=\alpha n\mathbf{x}_1+\beta n\mathbf{x}_2=n(\alpha\mathbf{x}_1+\beta\mathbf{x}_2),$$

i.e. any direction in the plane $\mathbf{x}_1$, $\mathbf{x}_2$ is again a principal axis with the same value n. Similarly it will be proved that if three non-coplanar principal axes have the same values n, any direction in space is again a principal axis with the same value of n. In this case the operator ω degenerates into a simple numerical factor or multiplier of the operand.

Again, if the principal values corresponding to $\mathbf{x}_1$, $\mathbf{x}_2$ are different, $n_1\neq n_2$, there is no third principal axis coplanar with $\mathbf{x}_1$, $\mathbf{x}_2$. For if $\mathbf{r}=\lambda\mathbf{x}_1+\mu\mathbf{x}_2$ represented such an axis, we should have

$$\omega\mathbf{r}=n(\lambda\mathbf{x}_1+\mu\mathbf{x}_2)=\lambda n_1\mathbf{x}_1+\mu n_2\mathbf{x}_2,$$

which would require n to be equal to both n_1 and n_2, contradicting the assumption.

Thus, if $n_1\neq n_2$, a third principal axis must be non-coplanar with the first two. Let $\mathbf{x}_3$ be such an axis and let $\omega\mathbf{x}_3=n_3\mathbf{x}_3$. Then, if $\mathbf{r}=\lambda\mathbf{x}_1+\mu\mathbf{x}_2+\nu\mathbf{x}_3$ were a fourth principal axis and n the corresponding principal value, we should have

$$n(\lambda\mathbf{x}_1+\mu\mathbf{x}_2+\nu\mathbf{x}_3)=\lambda n_1\mathbf{x}_1+\mu n_2\mathbf{x}_2+\nu n_3\mathbf{x}_3,$$

which is only possible for $n=n_1=n_2=n_3$, contrary to the assumption.

Thus, either n_1, n_2, n_3 are all different, when no fourth principal axis is possible, or two of these values coincide, when there is a whole flat pencil of principal axes, or all three values are equal, when every direction in space is such an axis.

In fine, a linear vector operator has at the utmost three essentially different principal axes.

It does not follow, of course, that every such operator has even as many as three real principal axes. It may have but one real axis (and two imaginary ones) notwithstanding that all the nine constituents $\omega_{\iota\kappa}$ are real. Only the self-conjugate or symmetrical operator, which may be characterized by $\omega_{\iota\kappa}=\omega_{\kappa\iota}$, has always three real principal axes.

The *symmetrical* operator can be defined, without the aid of a reference frame (implied by the $\omega_{\iota\kappa}$), as follows. If $\mathbf{A}$ and $\mathbf{B}$ are any two vectors, then in general $\mathbf{A}\omega\mathbf{B}$, the scalar product of $\mathbf{A}$ into $\omega\mathbf{B}$, is different from $\mathbf{B}\omega\mathbf{A}$. In particular, if

$$\mathbf{A}\omega\mathbf{B}=\mathbf{B}\omega\mathbf{A}, \qquad (S)$$

for any $\mathbf{A}$, $\mathbf{B}$, then ω is called a self-conjugate or symmetrical operator. The *orthogonality* of the principal axes of such an operator follows at once from this definition. In fact, if $\mathbf{x}$ and $\mathbf{y}$ be two such axes with the principal values n_1 and n_2, we have

$$\omega\mathbf{x}=n_1\mathbf{x}, \quad \omega\mathbf{y}=n_2\mathbf{y},$$

whence, multiplying the first of these equations by $\mathbf{y}$ and the second by $\mathbf{x}$,

$$(n_1-n_2)\mathbf{xy}=\mathbf{y}\omega\mathbf{x}-\mathbf{x}\omega\mathbf{y}=0.$$

Thus, either $n_1=n_2$, when, as we already know, every direction coplanar with $\mathbf{x}$, $\mathbf{y}$ is an axis, or $n_1\neq n_2$, when $\mathbf{xy}=0$ or $\mathbf{x}\perp\mathbf{y}$.

Q.E.D.

The cubic for the principal values of an operator ω, symmetrical or not, can be most easily obtained as follows. As before, let the operator be defined by $\omega\mathbf{a}=\mathbf{a}'$, $\omega\mathbf{b}=\mathbf{b}'$, $\omega\mathbf{c}=\mathbf{c}'$. Then it can be shown without trouble that, if n be a principal value, the three vectors

$$\mathbf{a}'-n\mathbf{a}, \quad \mathbf{b}'-n\mathbf{b}, \quad \mathbf{c}'-n\mathbf{c}$$

are coplanar. Such being the case, we have the equation

$$(\mathbf{a}'-n\mathbf{a})\mathrm{V}(\mathbf{b}'-n\mathbf{b})(\mathbf{c}'-n\mathbf{c})=0,$$

which is already the required cubic for n. Here $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is any triad of non-coplanar vectors. In particular, if this is a normal system of unit vectors, so that $\mathbf{a}=\mathrm{V}\mathbf{bc}$, $\mathbf{a}\mathrm{V}\mathbf{bc}=1$, etc., the cubic becomes

$$n^3-g_2n^2+g_1n-g_0=0,$$

where

$$g_0=\mathbf{a}'\mathrm{V}\mathbf{b}'\mathbf{c}', \quad g_1=\mathbf{a}\mathrm{V}\mathbf{b}'\mathbf{c}'+\mathbf{b}\mathrm{V}\mathbf{c}'\mathbf{a}'+\mathbf{c}\mathrm{V}\mathbf{a}'\mathbf{b}',$$

$$g_2=\mathbf{aa}'+\mathbf{bb}'+\mathbf{cc}'.$$

This proves also the invariance of these expressions for g_0, g_1, g_2, that is to say, their independence of the choice of the reference triad $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. For the principal values of ω, the roots of the cubic, are manifestly independent of any such framework of reference.

If, for any $\mathbf{A}$, $\mathbf{B}$,

$$\mathbf{A}\omega\mathbf{B}=\mathbf{B}\omega'\mathbf{A},$$

the operator ω' is said to be *the conjugate* of ω. This is preferable to the definition $\omega_{\iota\kappa}=\omega'_{\kappa\iota}$ used in Chapter V.

An *antisymmetrical* or *skew* operator ω can most directly be defined by the equation

$$\mathbf{A}\omega\mathbf{B}=-\mathbf{B}\omega\mathbf{A}, \qquad (A)$$

to hold for any $\mathbf{A}$, $\mathbf{B}$. This implies $\mathbf{A}\omega\mathbf{A}=0$, for any $\mathbf{A}$. The proof that any linear vector operator can be represented by a sum of a symmetrical and a skew operator now becomes immediate. In fact, if ω' be the conjugate of the given operator ω, we can write

$$\omega=\tfrac{1}{2}(\omega+\omega')+\tfrac{1}{2}(\omega-\omega').$$

Now, the first part of this operator is, by the definition (S), symmetrical and the second, by (A), antisymmetrical. In fact,

$$\mathbf{A}(\omega+\omega')\mathbf{B}=\mathbf{B}\omega'\mathbf{A}+\mathbf{B}\omega\mathbf{A}=\mathbf{B}(\omega+\omega')\mathbf{A},$$

and

$$\mathbf{A}(\omega-\omega')\mathbf{B}=\mathbf{B}\omega'\mathbf{A}-\mathbf{B}\omega\mathbf{A}=-\mathbf{B}(\omega-\omega')\mathbf{A}.$$

E. Euler's Angles.

Euler's angles, mentioned on p. 87, *i.e.* the three independent parameters determining the orientation of a rigid frame free to rotate around a fixed point, can be most easily introduced by the following vectorial treatment.

Let $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ and $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be two coinitial right-handed systems of unit vectors, the first fixed or to serve as a reference system, and the second rigidly attached to the frame which can be turned around the origin of these vectors. It is required to determine the orientation of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ relatively to $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ by means of three independent angles.

First, introduce the angle of *inclination* of $\mathbf{c}$ to $\mathbf{k}$,

$$i = (\mathbf{c}, \mathbf{k}),$$

so that $\mathbf{ck} = \cos i$. This will be one of Euler's angles, commonly denoted by θ. In the next place define an auxiliary unit vector $\mathbf{q}$ by

$$\sin i \,.\, \mathbf{q} = \mathrm{V}\mathbf{kc},$$

so that, in astronomical language, $\mathbf{q}$ will lie along *the line of nodes*, the intersection of the planes $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{i}$, $\mathbf{j}$. Then introduce the two independent angles,

$$f = (\mathbf{a}, \mathbf{q}),$$

which has no particular name, and *the longitude of the node*

$$\Omega = (\mathbf{q}, \mathbf{i}).$$

These will be the remaining two Eulerian angles, commonly denoted by ϕ and by ψ respectively.

It remains to deduce the formulae for the direction cosines $a_1 = \mathbf{ai}$, $a_2 = \mathbf{aj}$, etc., $c_3 = \mathbf{ck}$ in terms of the three angles i, f, Ω; in other words, to express the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ as linear functions of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ with coefficients containing only i, f, Ω.

Now, drawing the auxiliary unit vector $\mathbf{d}$ coplanar with $\mathbf{i}$, $\mathbf{j}$ and perpendicular to $\mathbf{q}$, *i.e.*

$$\mathbf{d} = \mathbf{i} \sin \Omega - \mathbf{j} \cos \Omega,$$

we have

$$\mathbf{c} = \mathbf{k} \cos i + \mathbf{d} \sin i = \sin i \,[\mathbf{i} \sin \Omega - \mathbf{j} \cos \Omega] + \mathbf{k} \cos i,$$

which is already one of the required vector equations, embodying the three usual cosine formulae

$$c_1 = \cos(\mathbf{c}, \mathbf{i}) = \sin i \sin \Omega, \quad c_2 = -\sin i \cos \Omega, \quad c_3 = \cos i.$$

Next, to find a similar equation for $\mathbf{a}$, notice that both $\mathbf{a}$ and $\mathbf{q}$ are normal to $\mathbf{c}$ and that the angle between $\mathbf{a}$ and $\mathbf{q}$ is f, so that

$$\mathrm{V}\mathbf{aq} = \mathbf{c} \sin f, \quad \mathbf{aq} = \cos f.$$

Whence

$$\mathrm{V}\mathbf{q}\mathrm{V}\mathbf{aq} = \mathbf{a} - \mathbf{q} \cos f = \sin f \,\mathrm{V}\mathbf{qc} = \frac{\sin f}{\sin i} [\mathbf{c} \cos i - \mathbf{k}],$$

and substituting here the value for $\mathbf{c}$ just found,

$$\mathbf{a} = \mathbf{q} \cos f + \sin f \,[(\mathbf{i} \sin \Omega - \mathbf{j} \cos \Omega) \cos i - \mathbf{k} \sin i],$$

where $\mathbf{q} = \mathbf{i} \cos \Omega + \mathbf{j} \sin \Omega$, as will be seen at once from the definition of this vector. Having thus found $\mathbf{a}$ and $\mathbf{c}$ we need not write $\mathbf{b}$ explicitly. For the latter is simply $\mathrm{V}\mathbf{ca}$.

Thus, collecting the scattered results, we have the required Eulerian expressions for the nine direction cosines $a_1 = \mathbf{ai}$, etc., $c_3 = \mathbf{ck}$, condensed in the three vector formulae

$$\begin{aligned} \mathbf{a} = \mathbf{i} &[\cos f \cos \Omega + \sin f \sin \Omega \cos i] \\ &+ \mathbf{j} [\cos f \sin \Omega - \sin f \cos \Omega \cos i] \\ &- \mathbf{k} \sin f \sin i, \end{aligned}$$

$$\mathbf{c} = [\mathbf{i} \sin \Omega - \mathbf{j} \cos \Omega] \sin i + \mathbf{k} \cos i,$$

$$\mathbf{b} = \mathrm{V}\mathbf{ca}.$$

With these values of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ we have for the components of the spin $\mathbf{p}$ of the rigid frame taken along $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, as in (47), p. 89, the formulae

$$p_a = \dot{\mathbf{b}}\mathbf{c} = -\mathbf{b}\dot{\mathbf{c}}, \text{ etc.},$$

which are identical with the known Eulerian formulae, as the reader will easily verify by expanding these expressions.

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CORRIGENDA

Page 5, line 11 from bottom, *for* consequent *read* consistent.
Page 12, line 1 from top, *for* colotropic *read* acolotropic.
Page 62, line 1 from top, *for* must *read* need.
Page 100, line 8 from top, *for* has *read* have.
Page 163, line 3 from top, *for* (93*b*) *read* (93*a*).
Page 177, line 19 from top, *for* peculiarly *read* particularly.
Page 178, line 2 from top, *for* atom *read* old-fashioned atom.

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